Perla Sousi ⟨ps422@cam.ac.uk⟩ *Michaelmas 2021*

1. Let X be a Markov chain containing an absorbing state s to which all other states lead (i.e., $j \to s$ for all j). Show that all states other than s are transient.

2. Compute $p_{11}^{(n)}$ and classify the states of the Markov chain with state space $I = \{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}.$$

3. A particle performs a random walk on the vertices of a cube. At each step it remains where it is with probability $\frac{1}{4}$, and moves to each of its neighbouring vertices with probability $\frac{1}{4}$. Let v and w be two diametrically opposite vertices. If the walk starts at v, find (a) the mean number of steps until its first return to v, (b) the mean number of steps until its first visit to w, (c) the mean number of visits to w before its first return to v.

4. (Harder) Let X be a Markov chain on $\{0, 1, 2, \dots\}$ with transition matrix given by $p_{0,j} = a_j$ for $j \geq 0$, $p_{i,i} = r$ and $p_{i,i-1} = 1 - r$ for $i \geq 1$. Assume that $0 < r < 1$. Classify the states of the chain, and find their mean recurrence times. [You may find it useful to define $J = \sup\{j : a_j > 0\}$.]

5. In Exercise 1.10, which states are recurrent and which are transient?

6. What can be said about the number of visits to each state in the case where (a) a Markov chain is transient, and (b) a Markov chain is recurrent?

Consider the Markov chain $(X_n)_{n\geq 0}$ of Exercise 1.13. Show for this chain that $\mathbb{P}[X_n \to \infty \text{ as } n \to \infty] = \mathbb{P}[\forall m, \exists n \text{ such that } X_N \geq m \text{ for all } N \geq n] = 1$.

Suppose the transition probabilities satisfy instead

$$p_{i,i+1} = \left(\frac{i+1}{i}\right)^{\alpha} p_{i,i-1}.$$

For each $\alpha \in (0, \infty)$ find the value of $\mathbb{P}[X_n \to \infty \text{ as } n \to \infty]$.

7. The rooted binary tree is an infinite graph T with one distinguished vertex R from which comes a single edge; at every other vertex there are three edges and there are no closed loops. The random walk on T jumps from a vertex along each available edge with equal probability. Show that the random walk is transient.

8. Show (by projection onto $\mathbb{Z}^3$ or otherwise) that the simple symmetric random walk in $\mathbb{Z}^4$ is transient.

9. Find all invariant distribution of the transition matrix in Exercise 1.10.

10. Two containers A and B are placed adjacently to one another, and gas is allowed to pass through a small aperture joining them. There are N molecules in all, and we assume that, at each epoch of time, one molecule (chosen at random) passes through the aperture. Show that the number of molecules in A evolves as a Markov chain. What are the transition probabilities? What is the invariant distribution of this chain? [This is the 'Ehrenfest urn model', first introduced by Ehrenfest under the name 'dog–flea model'.]

11. (Optional) A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}[X_n \text{ is a multiple of } 13]$$

quoting carefully any general theorems that you use.

12. Find the invariant distributions of the transition matrices in Exercise 1.9, parts (a), (b) and (c), and compare them with your answers to that exercise.

13. Each morning a student takes one of the three books (labelled 1, 2, 3) he owns from his shelf. The probability that he chooses the book with label i is α_i (where $0 < \alpha_i < 1$, $i = 1, 2, 3$), and choices on successive days are independent. In the evening he replaces the book at the left-hand end of the shelf. If p_n denotes the probability that on day n the student finds the books in the order 1,2,3, from left to right, show that, irrespective of the initial arrangement of the books, p_n converges as $n \to \infty$, and determine the limit.

14. In each of the following cases determine whether the stochastic matrix P corresponds to a chain which is reversible in equilibrium:

(a) $\begin{pmatrix} p & 1-p \\ q & 1-q \end{pmatrix}$;

(b) $\begin{pmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{pmatrix}$;

(c) $I = \{0, 1, 2, \ldots\}$ and $p_{01} = 1$, $p_{i,i+1} = p$, $p_{i,i-1} = 1 - p$ for $i \geq 1$.

(d) $p_{ij} = p_{ji}$ for all $i, j \in I$.

15. A random walk on the set $\{0, 1, 2, \ldots, b\}$ has transition matrix given by $p_{00} = 1 - \lambda_0$, $p_{bb} = 1 - \mu_b$, $p_{i,i+1} = \lambda_i$ and $p_{i+1,i} = \mu_{i+1}$ for $0 \leq i < b$, where $0 < \lambda_i, \mu_i < 1$ for all i, and $\lambda_i + \mu_i = 1$ for $1 \leq i < b$. Show that this process is time-reversible in equilibrium.

16. Let X be an irreducible non-null recurrent aperiodic Markov chain. Show that X is time-reversible in equilibrium if and only if

$$p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{n-1} j_n} p_{j_n j_1} = p_{j_1 j_n} p_{j_n j_{n-1}} \cdots p_{j_2 j_1}$$

for all n and all finite sequences $j_1, j_2, \ldots, j_n$ of states.