


Lecture 8

Dual spaces and dual maps

Df V F vector space

$V^* = \text{dual of } V$

$$= L(V, F) = \{ \alpha : V \rightarrow F \text{ linear} \}$$

Notation $\alpha : V \rightarrow F$ linear

$\equiv$ linear form

Ex (i) $\text{Tr} : M_{n,n}(F) \rightarrow F_n$

$$A = (a_{ij}) \mapsto \sum_{i=1}^n a_{ii}$$

$$\text{Tr} \in M_{n,n}^*$$

(ii) $f : [0,1] \rightarrow \mathbb{R}$

$$x \mapsto f(x)$$

$$Tf : C^\infty([a, b], \mathbb{R}) \longrightarrow \mathbb{R}$$

$$f \longmapsto \int_a^b f(x) \varphi(x) dx$$

If linear form on $C^\infty([a, b], \mathbb{R})$

$\leadsto$ you can construct f from Tf .

Lemma (Def) let V be a vector space over F ,
with a finite basis

$$B = \{e_1, \dots, e_n\}$$

Then there exists a basis for V^* given by :

$$B^* = \{\epsilon_1, \dots, \epsilon_n\} \quad \text{where:}$$

$$\epsilon_j \left(\sum_{i=1}^n a_i e_i \right) = a_j, \quad 1 \leq j \leq n$$

B^* = dual basis of B .

Remark Kronecker symbol :

$$S_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$e_j \left(\sum_{i=1}^n a_i e_i \right) = a_j$$

$$\Leftrightarrow \boxed{e_j(e_i) = S_{ij}} \leftarrow$$

proof . $(e_1, \dots, e_n)$ free.

$$\sum_{j=1}^n \lambda_j e_j = 0 \Rightarrow \sum_{j=1}^n \lambda_j \overbrace{e_j(e_i)}^{S_{ij}} = 0$$

$\forall i \in \{1, \dots, n\}$

$$\Rightarrow \lambda_i = 0, \forall 1 \leq i \leq n.$$

Span $\alpha \in V^*$,

$$\alpha(x) = \alpha\left(\sum_{j=1}^n \lambda_j e_j\right) = \sum_{j=1}^n \lambda_j \alpha(e_j)$$

On the other hand, let: $\sum_{j=1}^n \alpha(e_j) e_j \in V^*$,

$$\sum_{j=1}^n \alpha(e_j) e_j(x) = \sum_{j=1}^n \alpha(e_j) e_j\left(\sum_{k=1}^n \lambda_k e_k\right)$$

$$= \sum_{j=1}^n \alpha(e_j) \sum_{k=1}^n \underbrace{e_j(e_k)}_{\delta_{jk}} \lambda_k$$

$$= \sum_{j=1}^n \alpha(e_j) \lambda_j = \alpha(x)$$

$$\Rightarrow \alpha = \sum_{j=1}^n \alpha(e_j) e_j$$

□

Cor V finite dimensional: $\dim V^* = \dim V$

Remark It is sometimes convenient to think of V^* as the space of row vectors of length n over F .

$$e_1, \dots, e_n \text{ basis of } V, \quad x = \sum x_i e_i \in V$$

$$e_1, \dots, e_n \text{ basis of } V^*, \quad \alpha = \sum \alpha_i e_i \in V^*$$

$$\begin{aligned} \alpha(x) &= \sum_{i=1}^n \alpha_i e_i \left(\sum_{j=1}^n x_j e_j \right) \\ &= \sum_{i,j} \alpha_i x_j \underbrace{e_i(e_j)}_{\delta_{ij}} = \sum_{i=1}^n \alpha_i x_i \\ &= \begin{pmatrix} \alpha_1 & \dots & \alpha_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n \alpha_i x_i \end{aligned}$$

→ scalar product structure.

Def If $U \subseteq V$, the annihilator of U is:

subset $\nearrow$

$$U^\circ = \{ \alpha \in V^* / \forall u \in U, \alpha(u) = 0 \}$$

Lemma (i) $U^\circ \leq V^*$ (vector subspace)

(ii) If $U \leq V$ and $\dim V < +\infty$,

$$\dim V = \dim U + \dim U^\circ$$

proof (i). $0 \in U^\circ$

If $\alpha, \alpha' \in U^\circ$ then:

$$\forall u \in U, (\alpha + \alpha')(u) = \underbrace{\alpha(u)}_0 + \underbrace{\alpha'(u)}_0 = 0$$

and: $\forall \lambda \in F, (\lambda \alpha)(u) = \lambda \alpha(u) = 0$
 $\Rightarrow U^\circ \leq V^*$

(ii) let $U \leq V$, $\dim V = n$. let
 $(e_1, \dots, e_k)$ basis of U , complete it
to a basis $(\underbrace{e_1, \dots, e_k, e_{k+1}, \dots, e_n}_B)$ of V .

let $(\underbrace{\varepsilon_1, \dots, \varepsilon_n}_{B^*})$ be the dual basis of B

Claim $U^\circ = \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle$

. If $i > k$, $\varepsilon_i(e_k) = \delta_{ik} = 0$
 $\Rightarrow \varepsilon_i \in U^\circ \quad (U = \langle e_1, \dots, e_k \rangle)$
 $\Rightarrow \langle \varepsilon_{k+1}, \dots, \varepsilon_n \rangle \subset U^\circ$

• let $\alpha \in U^\circ$ then: $\alpha = \sum_{i=1}^n d_i e_i$
(B^* basis of V^*)

Now for $i \leq k$:

$$\alpha \in U^\circ \Rightarrow \alpha(e_i) = 0$$

$$\Rightarrow \alpha(e_i) = \sum_{j=1}^n d_j \underbrace{e_j(e_i)}_{\delta_{ij}} = d_i$$

$$\Rightarrow d_i = 0, \quad \forall 1 \leq i \leq k$$

$$\Rightarrow \alpha = \sum_{i=k+1}^n d_i e_i$$

$$\boxed{e_j(e_i) = \delta_{ij}}$$

$$\Rightarrow \alpha \in \langle e_{k+1}, \dots, e_n \rangle$$

$$\Rightarrow U^\circ \subset \langle e_{k+1}, \dots, e_n \rangle \quad \square$$

Def / Lemma

V, W vector spaces over F . Let $\alpha \in L(V, W)$. Then the

map: $\alpha^* : W^* \longrightarrow V^*$ is

$$\varepsilon \longmapsto \varepsilon \circ \alpha$$

an element of $L(W^*, V^*)$. It is called the dual map of α .

proof. $\varepsilon \circ \alpha : V \longrightarrow F$ linear by linearity of ε and α

$$\Rightarrow \varepsilon \circ \alpha \in V^*$$

α^* linear: pick $\partial_1, \partial_2 \in W^*$, then:

$$\begin{aligned} \alpha^*(\partial_1 + \partial_2) &= (\partial_1 + \partial_2) \circ \alpha = \partial_1 \circ \alpha + \partial_2 \circ \alpha \\ &\stackrel{\text{def}}{=} \alpha^*(\partial_1) + \alpha^*(\partial_2) \end{aligned}$$

and similarly: $\forall \lambda \in \overline{F}$

$$\alpha^*(\lambda \theta) = \lambda \alpha^*(\theta)$$

$$\Rightarrow \alpha^* \text{ linear, } \alpha^* \in L(W^*, V^*)$$

Prop

Let V, W finite dimensional spaces over F with basis $\underline{B}, \underline{C}$. Let $\underline{B}^*, \underline{C}^*$ be the dual basis of V^*, W^* then:

$$[\alpha^*]_{\underline{C}^*, \underline{B}^*} = [\alpha]_{\underline{B}, \underline{C}}^T$$

Dual map $\Leftrightarrow$ Adjoint of α

proof

$$\underline{B} = (b_1, \dots, b_n), \underline{C} = (c_1, \dots, c_m)$$

$$\underline{B}^* = (\beta_1, \dots, \beta_n), \underline{C}^* = (\gamma_1, \dots, \gamma_m)$$

Say: $[\alpha]_{\mathcal{B}_V \mathcal{B}_W} = A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \leftarrow$

Let's compute: $(\alpha^*: W^* \rightarrow V^*)$

$$\alpha^*(\gamma_r)(b_s)$$

$$= \gamma_r \circ \alpha(b_s) = \gamma_r(\overbrace{\alpha(b_s)}^w)$$

$$= \gamma_r\left(\underbrace{\sum_t a_{ts} c_t}_{s\text{-th column vector}}\right) = \sum_t a_{ts} \underbrace{\gamma_r(c_t)}_{S_{rt}} = a_{rs}$$

$$A = [\alpha]_{\mathcal{B}_V \mathcal{B}_W} = \left(\alpha(b_1) \mid \dots \mid \alpha(b_n) \right) \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

We have computed :

$$\alpha^*(\gamma_r)(b_s) = a_{rs}$$

On the other hand :

$$\underbrace{\left(\sum_i d_{ri} \beta_i \right)}_{\in V^*} \underbrace{(b_s)}_{\in V} = \sum_i d_{ri} \overbrace{\beta_i(b_s)}^{S_{is}} = a_{rs}$$

$$\Rightarrow \forall s, \quad \alpha^*(\gamma_r)(b_s) = \left(\sum_i d_{ri} \beta_i \right)(b_s)$$

$$\Rightarrow \underbrace{\alpha^*(\gamma_r)}_{\in V^*} = \sum_i \underbrace{d_{ri}}_{\in \mathbb{R}} \beta_i$$

$$\left(\alpha^*(\gamma_1) \mid \dots \mid \alpha^*(\gamma_m) \right) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha^* \end{bmatrix}_{\mathcal{C}, \mathcal{B}^*} = A^T$$

②