


Lecture 23

Spectral theory for self adjoint maps

Spectral theory $\equiv$ study of the spectrum of operators

→ mathematics

→ physics : QUANTUM MECHANICS

→ infinite dimension (Hilbert spaces)

Finite dimension $\rightarrow$ infinite dimension
(linear operators $\rightarrow$ compact operators)

Adjoint operator V, W finite dimensional inner product spaces, $\alpha \in L(V, W)$

then the adjoint $\alpha^* \in L(W, V)$ such that

$$\forall (v, w) \in V \times W, \quad \langle \alpha(v), w \rangle = \langle v, \alpha^*(w) \rangle.$$

We defined :

- self adjoint maps $\alpha = \alpha^*$ ($V = W$)

$$\Leftrightarrow \forall (v, w) \in V \times V, \\ \langle \alpha v, w \rangle = \langle v, \alpha w \rangle$$

- isometries $V = W, \alpha^* = \alpha^{-1}$

$$\Leftrightarrow \forall (v, w) \in V \times V, \langle \alpha v, \alpha w \rangle = \langle v, w \rangle$$

• $\mathbb{R}$: orthogonal group

• $\mathbb{C}$: unitary group

Spectral theory for self adjoint operators

lemma let V be a finite dimensional inner product space. let $\alpha \in L(V)$ be

self adjoint $\alpha = \alpha^*$. Then:

(i) α has real eigenvalues

(ii) eigenvectors of α with respect to different eigenvalues are orthogonal.

proof (i) $v \in V \setminus \{0\}$, $\lambda \in \mathbb{C}$ /
 $\alpha v = \lambda v$

$$\begin{aligned} \Rightarrow \langle \lambda v, v \rangle &= \lambda \|v\|^2 = \langle \alpha v, v \rangle \\ &= \langle v, \alpha^* v \rangle = \langle v, \alpha v \rangle \\ &= \langle v, \lambda v \rangle = \overline{\lambda} \|v\|^2 \end{aligned}$$

↑
antilinear

$$\begin{aligned} \Rightarrow (\lambda - \overline{\lambda}) \underbrace{\|v\|^2}_{\neq 0} &= 0 \Rightarrow \lambda = \overline{\lambda} \\ &\Rightarrow \lambda \in \mathbb{R} \end{aligned}$$

$$\begin{array}{l|l} \text{(ii)} & \alpha v = \lambda v \quad \lambda \in \mathbb{R}, v \neq 0 \\ & \alpha w = \mu w \quad \mu \in \mathbb{R}, w \neq 0 \end{array}$$

$$\begin{aligned} \text{Then : } \lambda \langle v, w \rangle &= \langle \alpha v, w \rangle \\ &= \langle v, \alpha^* w \rangle = \langle v, \mu w \rangle \\ &= \mu \langle v, w \rangle \end{aligned}$$

$$\Rightarrow \underbrace{(\lambda - \mu)}_{\neq 0} \langle v, w \rangle = 0$$

$$\Rightarrow \langle v, w \rangle = 0$$

0.

Thm let V be a finite dimensional inner product space. let $\alpha \in L(V)$ be self adjoint. Then V has an orthonormal basis of eigenvectors of α .

→ We also say: α can be diagonalized in an orthonormal basis for V .

proof $F = \mathbb{R} \text{ or } \mathbb{C}$. We argue by induction on the dimension of V .

• $n = 1$: ✓.

• $n - 1 \mapsto n$. Say $A = [\alpha]_B$ wrt standard basis B .

By the fundamental theorem of algebra, we know that $\chi_A(\lambda)$ ($\equiv$ characteristic polynomial) has a (complex) root. This root is an eigenvalue of $\alpha = \alpha^* \Rightarrow$ this root is real.

Let us call this real eigenvalue $\lambda \in \mathbb{R}$

pick $\begin{cases} v_1 \in V \setminus \{0\} \\ \|v_1\| = 1 \end{cases}, \quad \alpha v_1 = \lambda v_1$

let : $U = \langle v_1 \rangle^{\perp} \leq V$.

key observation U stable by α : $\alpha(U) \leq U$.

Indeed, let $u \in U$,

$$\begin{aligned} \langle \alpha u, v_1 \rangle &= \langle u, \alpha^* v_1 \rangle = \langle u, \alpha v_1 \rangle \\ &= \langle u, \rangle v_1 \rangle = \langle u, v_1 \rangle = 0 \end{aligned}$$

$\uparrow$
 $u \in U = \langle v_1 \rangle^{\perp}$

$$\Rightarrow \langle \alpha u, v_1 \rangle = 0 \Rightarrow \alpha u \in U.$$

. This implies : we may consider $\alpha|_U$ which is also self adjoint, and :

$$n = \dim V = \dim U + 1$$

$$\Rightarrow \dim U \leq n - 1$$

$$\Rightarrow \exists (v_1, \dots, v_n) \text{ orthonormal basis of eigenvectors of } \alpha|_U$$

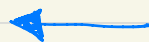
$\uparrow$
induction on n

$\Rightarrow$
 $\uparrow$
 $U = \langle u_1 \rangle^\perp$

$(u_1, u_2, \dots, u_n)$ orthonormal basis of V .

Rk

$$A = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & \boxed{\begin{matrix} \ddots \\ \vdots \end{matrix}} & & \\ \vdots & & \ddots & \\ 0 & & & \lambda \end{pmatrix} \begin{matrix} u_1 \\ \vdots \\ u_n \end{matrix}$$



Corollary

V finite dimensional inner product space.

If $\alpha \in L(V)$ is self adjoint

then V is the orthogonal direct sum of all the eigenspaces of α .

Spectral theory for unitary maps

Lemma V be a complex inner product space.
(Hermitian sesquilinear structure). let
 $\alpha \in L(V)$ be unitary ($\alpha^* = \alpha^{-1}$).

Then :

- (i) all eigenvalues of α lie on the unit circle
- (ii) eigenvectors corresponding to distinct eigenvalues are orthogonal.

proof (i) $\alpha v = \lambda v$ $\xleftarrow{\alpha^{-1}}$
 $|v \in V \setminus \{0\}, \lambda \in \mathbb{C}$

$\lambda \neq 0$ (α invertible)

$$\begin{aligned} \lambda \langle v, v \rangle &= \langle \alpha v, v \rangle = \langle v, \alpha^* v \rangle \\ &= \langle v, \alpha^{-1} v \rangle = \langle v, \lambda^{-1} v \rangle = \frac{1}{\lambda} \langle v, v \rangle \end{aligned}$$

$$\Rightarrow \lambda \|v\|^2 = \frac{1}{\lambda} \|v\|^2 \Rightarrow (\lambda^2 - 1) \underbrace{\|v\|^2}_{\neq 0} = 0$$

$$\Rightarrow |\lambda| = 1.$$

$$\text{(ii).} \quad \left| \begin{array}{l} \alpha v = \lambda v \\ \alpha w = \mu w \end{array} \right| \quad \left| \begin{array}{l} \lambda, \mu \neq 0 \\ \mu \neq \lambda \end{array} \right|$$

$$\begin{aligned} \text{Then : } \lambda \langle v, w \rangle &= \langle \alpha v, w \rangle = \langle v, \alpha^* w \rangle \\ &= \langle v, \bar{\mu} w \rangle = \langle v, \mu^{-1} w \rangle \\ &= \frac{1}{\mu} \langle v, w \rangle = \mu \langle v, w \rangle \end{aligned}$$

$$\uparrow$$

$$|\mu| = 1$$

$$\Rightarrow \underbrace{(\lambda - \mu)}_{\neq 0} \langle v, w \rangle = 0 \Rightarrow \langle v, w \rangle = 0.$$

0.

Thm V be a finite dimensional complex inner product space. let $\alpha \in L(V)$ be unitary ($\alpha^* = \bar{\alpha}$). Then V has an orthonormal basis consisting of eigenvectors of α .

Equivalently, a unitary on V Hermitian is diagonalizable in an orthonormal basis.

proof $A = [\alpha]_{\mathcal{B}}$, $\chi_A(\lambda)$ has a complex root $\Rightarrow \alpha$ has a complex eigenvalue.

Fix $v_1 \in V \setminus \{0\}$, $\alpha v_1 = \lambda v_1$,
 $\|v_1\| = 1$

let $U = \langle v_1 \rangle^\perp$, then $\alpha(U) \subseteq U$.

Indeed : $u \in U$

$$\begin{aligned} \langle \alpha u, v_1 \rangle &= \langle u, \alpha^* v_1 \rangle = \langle u, \alpha^{-1} v_1 \rangle \\ &= \langle u, \frac{1}{\lambda} v_1 \rangle = \frac{1}{\lambda} \langle u, v_1 \rangle = 0 \end{aligned}$$

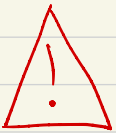
$\uparrow$
 $u \in U = \langle v_1 \rangle$

$\Rightarrow \alpha u \in U$

We consider $\alpha|_U$ which is unitary on U and diagonalizable in an orthonormal basis.

$(v_1, \dots, v_n)$ Thanks to the induction hypothesis.

Then $(v_1, \dots, v_n)$ is an orthonormal basis of V which diagonalizes α . □



We used the complex structure (to make sure that there is an eigenvalue (complex)).

In general, an real valued orthonormal

matrix A ($AA^T = Id$) cannot be diagonalized over $\mathbb{R}$.

Example Rotation in $\mathbb{R}^2$

$$A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

$$\chi_A(\lambda) = \begin{vmatrix} \cos \alpha - \lambda & -\sin \alpha \\ \sin \alpha & \cos \alpha - \lambda \end{vmatrix}$$

$$= (\cos \alpha - \lambda)^2 + \sin^2 \alpha$$

$$\Leftrightarrow \lambda = e^{\pm i\alpha} \quad \left(\notin \mathbb{R} \text{ for } \alpha = \frac{\pi}{2} \right)$$