Lecture 23 — Spectral theory for self adjoint maps

. **Spectral theory** ≡ study of the spectrum of $\boxed{\text{operators}}$

. → mathematics
→ physics : QUANTUM MECHANICS
⇝ infinite dimension (Hilbert spaces)

. Finite dimension → infinite dimension
(linear operators → Compact operators)

. **Adjoint operator** V, W finite dimensional inner product spaces, $\alpha \in L(V, W)$ then the adjoint $\alpha^* \in L(W, V)$ such that $\forall (v, w) \in V \times W$, $\langle \alpha(v), w \rangle = \langle v, \alpha^*(w) \rangle$.

We defined :

- self adjoint maps $\qquad \alpha = \alpha^{*} \qquad (V = W)$

$\iff \forall (v, w) \in V \times V,$

$$\langle \alpha v, w \rangle = \langle v, \alpha w \rangle$$

- isometries $\qquad V = W, \quad \alpha^{*} = \alpha^{-1}$

$\iff \forall (v, w) \in V \times V, \quad \langle \alpha v, \alpha w \rangle = \langle v, w \rangle$

- $\mathbb{R}$: orthogonal group.
- $\mathbb{C}$: unitary group.

Spectral theory for self adjoint operators

Lemma Let V be a finite dimensional inner product space. Let $\alpha \in L(V)$ be self adjoint $\alpha = \alpha^*$. Then:

<u>(i)</u> α has real eigenvalues

<u>(ii)</u> eigenvectors of α with respect to different eigenvalues are orthogonal.

<u>proof</u> <u>(i)</u> $v \in V \setminus \{0\}$, $\lambda \in \mathbb{C}$ /

$$\alpha v = \lambda v$$

$$\Rightarrow \langle \lambda v, v \rangle = \lambda \|v\|^2 = \langle \alpha v, v \rangle$$

$$= \langle v, \alpha^* v \rangle = \langle v, \alpha v \rangle$$

$$= \langle v, \underbrace{\lambda v}_{\text{antilinear}} \rangle = \bar{\lambda} \|v\|^2$$

$$\Rightarrow (\lambda - \bar{\lambda}) \underbrace{\|v\|^2}_{\neq 0} = 0 \Rightarrow \lambda = \bar{\lambda}$$

$$\Rightarrow \lambda \in \mathbb{R}$$

(iii) $\begin{cases} \alpha v = \lambda v & \lambda \in \mathbb{R},\ v \neq 0 \\ \alpha w = \mu w & \mu \in \mathbb{R},\ w \neq 0 \end{cases}$

Then: $\lambda \langle v, w \rangle = \langle \alpha v, w \rangle$

$$= \langle v, \alpha^* w \rangle = \langle v, \mu w \rangle$$

$$= \mu \langle v, w \rangle$$

$$\Rightarrow \underbrace{(\lambda - \mu)}_{\neq 0} \langle v, w \rangle = 0$$

$$\Rightarrow \langle v, w \rangle = 0$$

$\square$.

Thm Let V be a finite dimensional inner product space. Let $\alpha \in L(V)$ be self adjoint. Then V has an orthonormal basis of eigenvectors of α.

→ We also say: α can be diagonalized in an orthonormal basis for V.

proof $F = \mathbb{R}$ or $\mathbb{C}$. We argue by induction on the dimension of V.

- $n = 1$: ✓.
- $n-1 \mapsto n$. Say $A = [\alpha]_B$ wrt standard basis B.

By the fundamental theorem of algebra, we know that $\chi_A(\lambda)$ ($\equiv$ characteristic polynomial) has a (complex) root. This root is an eigenvalue of $\alpha = \alpha^*$ $\Longrightarrow$ this root is real.

Let us call this real eigenvalue $\lambda \in \mathbb{R}$

pick $v_1 \in V \setminus \{0\}$, $\alpha v_1 = \lambda v_1$.

$\|v_1\| = 1$

Let : $U = \langle v_1 \rangle^{\perp} \leq V$.

Key observation U stable by α : $\alpha(U) \leq U$.

Indeed let $u \in U$,

$$\langle \alpha u, v_1 \rangle = \langle u, \alpha^* v_1 \rangle = \langle u, \alpha v_1 \rangle$$
$$= \langle u, \lambda v_1 \rangle = \lambda \langle u, v_1 \rangle = 0$$

$u \in U = \langle v_1 \rangle^{\perp}$

$\Rightarrow \langle \alpha u, v_1 \rangle = 0 \Rightarrow \alpha u \in U$.

- This implies : we may consider $\alpha_{|U}$ which is also self adjoint, and :

$$n = \dim V = \dim U + 1$$
$$\Rightarrow \dim U \leq n - 1$$
$$\Rightarrow \exists (v_2, \dots, v_n) \text{ orthonormal basis of eigenvectors of } \alpha_{|U}$$

induction on n

$\Longrightarrow$ $(v_1, v_2, \dots, v_n)$ orthonormal basis of

$\uparrow$ $\underbrace{}_{U}$ V.

$U = \langle v_1 \rangle^{\perp}$

<u>Rk</u>

$$A = \begin{array}{c} \begin{array}{cccc} v_1 & & & \end{array} \\ \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & & & \ddots \end{pmatrix} \end{array} \begin{array}{c} v_1 \\ \wedge \\ \\ \vee \end{array}$$

Corollary V finite dimensional inner product space. If $\alpha \in L(V)$ is self adjoint then V is the orthogonal direct sum of all the eigenspaces of α.

Spectral theory for unitary maps

Lemma V be a complex inner product space (Hermitian sesquilinear structure). Let $\alpha \in L(V)$ be unitary $(\alpha^* = \alpha^{-1})$.

Then:

(i) all eigenvalues of α lie on the unit circle

(ii) eigenvectors corresponding to distinct eigenvalues are orthogonal.

proof (i) $\alpha v = \lambda v \leftarrow \alpha^{-1}$

$v \in V \setminus \{0\}$, $\lambda \in \mathbb{C}$

. $\lambda \neq 0$ (α invertible)

. $\lambda \langle v, v \rangle = \langle \alpha v, v \rangle = \langle v, \alpha^* v \rangle$

$= \langle v, \alpha^{-1} v \rangle = \langle v, \lambda^{-1} v \rangle = \frac{1}{\bar{\lambda}} \langle v, v \rangle$

$$\Rightarrow \lambda \|v\|^2 = \frac{1}{\bar\lambda}\|v\|^2 \Rightarrow (|\lambda|^2 - 1)\underbrace{\|v\|^2}_{\neq 0} = 0$$

$$\Rightarrow |\lambda| = 1.$$

(ii). $\left|\begin{array}{l} \alpha v = \lambda v \\ \alpha w = \mu w \end{array}\right. , \left|\begin{array}{l} \lambda, \mu \neq 0 \\ \mu \neq \lambda. \end{array}\right.$

Then: $\lambda \langle v, w\rangle = \langle \alpha v, w\rangle = \langle v, \alpha^* w\rangle$

$= \langle v, \alpha^{-1} w\rangle = \langle v, \mu^{-1} w\rangle$

$= \frac{1}{\bar\mu}\langle v, w\rangle \underset{|\mu| = 1}{=} \mu \langle v, w\rangle$

$$\Rightarrow \underbrace{(\lambda - \mu)}_{\neq 0}\langle v, w\rangle = 0 \Rightarrow \langle v, w\rangle = 0.$$

D.

Thm V be a finite dimensional complex inner product space. Let $\alpha \in L(V)$ be unitary ($\alpha^* = \alpha^{-1}$). Then V has an orthonormal basis consisting of eigenvectors of α.

Equivalently, a unitary on V Hermitian is diagonalizable in an orthonormal basis.

proof $A = [\alpha]_B$, $\chi_A(\lambda)$ has a complex root $\Rightarrow$ α has a complex eigenvalue

Fix $v_1 \in V \setminus \{0\}$, $\alpha v_1 = \lambda v_1$,
$\|v_1\| = 1$

Let: $U = \langle v_1 \rangle^\perp$, then $\alpha(U) \leq U$.

<u>Indeed</u> : $u \in U$

$$\langle \alpha u, v_1 \rangle = \langle u, \alpha^* v_1 \rangle = \langle u, \alpha^{-1} v_1 \rangle$$

$$= \langle u, \frac{1}{\lambda} v_1 \rangle = \frac{1}{\bar{\lambda}} \langle u, v_1 \rangle = 0$$

$u \in U = \langle v_1 \rangle^\perp$

$\Rightarrow \alpha u \in U$.

We consider $\alpha_{|U}$ which is unitary on U, and diagonalizable in an orthonormal basis $(v_2, \dots, v_n)$ thanks to the induction hypothesis.

Then $(v_1, \dots, v_n)$ is an orthonormal basis of V which diagonalizes α.

D.

⚠ We used the complex structure (to make sure that there is an eigenvalue (complex)).

In general, an real valued orthonormal

matrix A $(AA^T = Id)$ cannot be diagonalized over $\mathbb{R}$.

Example Rotation in $\mathbb{R}^2$

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\chi_A(\lambda) = \begin{vmatrix} \cos\alpha - \lambda & -\sin\alpha \\ \sin\alpha & \cos\alpha - \lambda \end{vmatrix}$$

$$= (\cos\alpha - \lambda)^2 + \sin^2\alpha$$

$$\iff \lambda = e^{\pm i\alpha} \quad \left(\notin \mathbb{R} \text{ for } \alpha = \frac{\pi}{2}\right)$$