


Lecture 21

Gram Schmidt and orthogonal complement

Def V vector space over $\mathbb{R}$ ($\mathbb{C}$). An inner product is a positive definite symmetric (Hermitian) form φ on V .

Notation $\langle u, v \rangle = \varphi(u, v)$

Norm $\|v\| = \sqrt{\langle v, v \rangle}$
 $\in \mathbb{R}_+$

and: $\|v\| = 0 \iff \langle v, v \rangle = 0$

$\iff v = 0$

$\uparrow$

definite

lemma (Cauchy-Schwarz)

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

proof $F = \mathbb{R} \text{ or } \mathbb{C}$. let $t \in F$, then:

$$0 \leq \underbrace{\|tu - v\|^2}_{\text{positive}} = \underbrace{\langle tu - v, tu - v \rangle}_{\substack{\uparrow \text{linear} \quad \uparrow \text{antilinear}}}$$

$$= t \bar{t} \langle u, u \rangle - t \langle u, v \rangle - \overline{t \langle v, u \rangle} + \langle v, v \rangle$$

$$= |t|^2 \|u\|^2 - 2 \operatorname{Re}(t \langle u, v \rangle) + \|v\|^2$$

We pick explicitly $t = \frac{\overline{\langle u, v \rangle}}{\|u\|^2}$ which gives:

$$0 \leq \frac{|\langle u, v \rangle|^2}{\|u\|^4} \|u\|^2 - 2 \operatorname{Re} \left(\frac{|\langle u, v \rangle|^2}{\|u\|^2} \right) + \|v\|^2$$

$$\Leftrightarrow 0 \leq \|v\|^2 - \frac{|\langle v, v \rangle|^2}{\|u\|^2}$$

$$\Leftrightarrow |\langle v, v \rangle|^2 \leq \|u\|^2 \|v\|^2$$

D.

Exercise Show that if there is equality in Cauchy Schwarz, then u, v are colinear.

Cor (Triangle inequality)

$$\|u+v\| \leq \|u\| + \|v\|$$

proof $\|u+v\|^2 = \langle u+v, u+v \rangle$

CS $= \|u\|^2 + 2\operatorname{Re}(\langle u, v \rangle) + \|v\|^2$

$\downarrow$
 $\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$ D.

Cor

$\|\cdot\|$ is a norm.

Def

A set $(e_1, \dots, e_n)$ of vectors of V is

- (i) ORTHOGONAL if $\langle e_i, e_j \rangle = 0$ for $i \neq j$
- (ii) ORTHONORMAL if $\langle e_i, e_j \rangle = \delta_{ij}$

$$\left(\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \right)$$

Lemma

If $(e_1, \dots, e_n)$ are orthogonal non zero vectors, then they are linearly independent.

However, if

$$v = \sum_{j=1}^n \lambda_j e_j, \quad \text{then:}$$

$$\lambda_j = \frac{\langle v, e_j \rangle}{\|e_j\|^2}$$

proof (i) $\sum_{i=1}^k \lambda_i e_i = 0$

$$\Rightarrow 0 = \left\langle \sum_{i=1}^k \lambda_i e_i, e_j \right\rangle$$

$$= \sum_{i=1}^k \lambda_i \langle e_i, e_j \rangle = \lambda_j \underbrace{\|e_j\|^2}_{\neq 0}$$

$$\Rightarrow \forall 1 \leq j \leq k, \lambda_j = 0.$$

(ii) $v = \sum_{i=1}^k \lambda_i e_i$

$$\Rightarrow \langle v, e_j \rangle = \lambda_j \|e_j\|^2$$

$$\Rightarrow \lambda_j = \frac{\langle v, e_j \rangle}{\|e_j\|^2}$$

Q.E.D.

Lemma (Parseval's identity)

If V is a finite dimensional inner product space and $(e_1, \dots, e_n)$ is an

ORTHONORMAL basis. Then:

$$\langle u, v \rangle = \sum_{i=1}^n \langle u, e_i \rangle \overline{\langle v, e_i \rangle}$$

proof

$$\begin{aligned} u &= \sum_{i=1}^n \langle u, e_i \rangle e_i \quad (\|e_i\| = 1) \\ v &= \sum_{i=1}^n \langle v, e_i \rangle e_i \end{aligned}$$

$$\begin{aligned} \Rightarrow \langle u, v \rangle &= \left\langle \sum_{i=1}^n \langle u, e_i \rangle e_i, \sum_{i=1}^n \langle v, e_i \rangle e_i \right\rangle \\ &= \sum_{i=1}^n \langle u, e_i \rangle \overline{\langle v, e_i \rangle} \end{aligned}$$

D.

In particular, in an orthonormal basis,

$$\|v\|^2 = \langle v, v \rangle = \sum_{i=1}^n |\langle v, e_i \rangle|^2$$

Theorem (GRAN-SCHMIDT orthogonalisation process)

V inner product space, $(v_i)_{i \in I}$, $v_i \in V$
 $(v_i)_{i \in I}$ linearly independent | I countable (or finite)

Then there exists a sequence $(e_i)_{i \in I}$ of
ORTHOGONAL vectors such that:

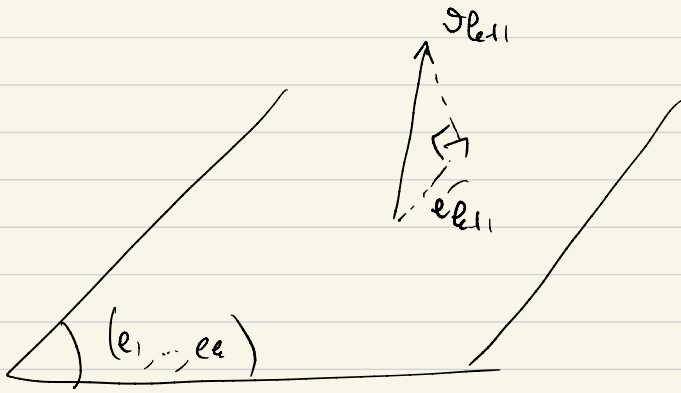
$$\begin{aligned} \text{Span} \langle v_1, \dots, v_k \rangle &= \text{Span} \langle e_1, \dots, e_k \rangle \\ \forall k \geq 1 \end{aligned}$$

proof. Induction on k

$$k=1, \quad e_1 = \frac{v_1}{\|v_1\|}$$

Say we found $(e_1, \dots, e_k)$. We look for e_{k+1} . Define:

$$e'_{k+1} = v_{k+1} - \sum_{i=1}^k \langle v_{k+1}, e_i \rangle e_i$$



• $e'_{k+1} \neq 0$: otherwise, $v_{k+1} \in \text{Span}\langle e_1, \dots, e_k \rangle$
 $= \text{Span}\langle v_1, \dots, v_k \rangle$ contradiction

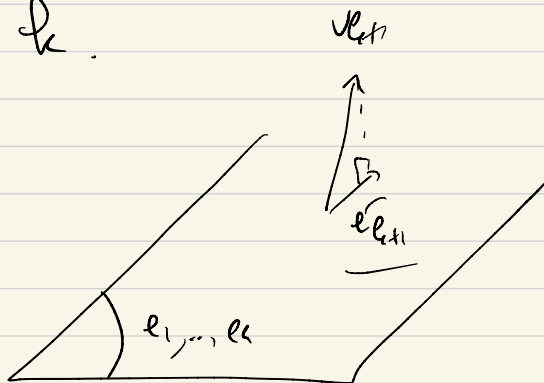
$$\begin{cases} \langle e'_{k+1}, e_j \rangle = \langle v_{k+1}, e_j \rangle - \langle v_{k+1}, e_j \rangle = 0 \\ 1 \leq j \leq k \end{cases}$$

$$\text{Span}\langle v_1, \dots, v_{k+1} \rangle = \text{Span}\langle e_1, \dots, e_k, e'_{k+1} \rangle$$

$\rightarrow e_{k+1} = \frac{e'_{k+1}}{\|e'_{k+1}\|}$ does the job

D.

$\Rightarrow$ This is an algorithm to compute $(e_1, \dots, e_k)$
for all k .



Corollary V finite dim, inner product space.
Any orthonormal set of vectors can
be extended to an orthonormal basis of V

proof Pick $(e_1, \dots, e_k)$ orthonormal. Then they
are linearly independent, so we can
extend $(e_1, \dots, e_k, v_{k+1}, \dots, v_n)$ basis of V .
Apply Gram Schmidt to this set (observe that

Here is need to modify $e_1, \dots, e_k$

$\Rightarrow (e_1, \dots, e_k, e_{k+1}, \dots, e_n)$ orthonormal basis of V .

Note $A \in M_{m,n}(\mathbb{R})$ has orthogonal
(C)

Columns if : $A^T A = \text{Id} \quad (\mathbb{R})$
 $A^T \bar{A} = \text{Id} \quad (\mathbb{C})$

Def $A \in M_n(\mathbb{R})$ is :

($\mathbb{R}$) - orthogonal if : $A^T A = \text{Id} \quad (\Leftrightarrow \bar{A}^t = A^T)$

($\mathbb{C}$) - unitary if : $A^T \bar{A} = \text{Id} \quad (\Leftrightarrow \bar{A}^t = \bar{A}^T)$

Prop $A \in M_n(\mathbb{R})$ ($[\mathbb{C}]$), non singular, then
A can be written : $A = R T$

Where:

- T upper triangular
- R orthogonal (unitary)

Proof Exercise: apply Gram Schmidt to the column vectors of A .

Orthogonal complement and projection

($\rightarrow$ natural and deep extension to the infinite dimensional setting: HILBERT spaces)

Def V inner product space, $V_1, V_2 \leq V$.

We say that V is the orthogonal direct sum of V_1 and V_2 if:

(i) $V = V_1 \oplus V_2$

(ii) $\forall (v_1, v_2) \in V_1 \times V_2, \langle v_1, v_2 \rangle = 0$

Notation $V = V_1 \overset{\perp}{\oplus} V_2$
 $\left(V = V_1 \overset{\perp}{\oplus} V_2 \right)$

Pr $v \in V_1 \cap V_2 \Rightarrow \langle v, v \rangle = \|v\|^2 = 0$
 (ii)
 $\Rightarrow v = 0$.

Def V inner product space

$W \leq V$, We define:

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}$$

Lemma V inner product space

$W \leq V$. Then:

$$V = W \overset{\perp}{\oplus} W^\perp$$

(*)

proof Observe that $W^\perp \leq V$.

$$\text{--- (ii) } \left. \begin{array}{l} w \in W \\ w^\perp \in W^\perp \end{array} \right\} ,$$

$$\langle w, w^\perp \rangle = 0 \quad \text{by definition of } W^\perp .$$

$$\text{Answer } w \in W \cap W^\perp \Rightarrow \|w\|^2 = \langle w, w \rangle = 0 \\ \Rightarrow w = 0 .$$

(i), We need to show that $V = W + W^\perp$

Let $(e_1, \dots, e_k)$ be an orthonormal basis of W .
Extend it to $(e_1, \dots, e_k, e_{k+1}, \dots, e_n)$
orthonormal basis of V . Observe that:

$$(e_{k+1}, \dots, e_n) \in W^\perp$$

$$\Rightarrow V = W + W^\perp$$