Lecture 13 Adjugate matrix

- **Observation** $A \in M_n(F)$

$$A = \left(A^{(1)} \mid \dots \mid A^{(n)} \right)$$

- Action of the determinant when swapping two column vectors: $1 \leq j < k \leq n$

$$\det \left(A^{(1)} \mid \dots \mid A^{(j)} \mid \dots \mid A^{(k)} \mid \dots \mid A^{(n)} \right)$$

$$= - \det \left(A^{(1)} \mid \dots \mid A^{(k)} \mid \dots \mid A^{(j)} \mid \dots \mid A^{(n)} \right)$$

↑
alternate n linear form

- Using that $\det A = \det(A^T)$, we can see that swapping two lines changes the determinant by a factor (-1).

Remark We could prove properties of determinant using the decomposition of A into elementary matrices.

Column (line) expansion and adjugate matrix

Column expansion ≡ to reduce the computation of $n \times n$ determinants to $(n-1) \times (n-1)$ determinants.

$\rightsquigarrow$ very useful to compute determinants

Def $A \in M_n(F)$

Pick i, j, $\quad \left| \begin{array}{l} i \in \{1, \dots, n\} \\ j \in \{1, \dots, n\} \end{array} \right.$

We define: $A_{\hat{i}\hat{j}} \in M_{n-1}(F)$ obtained by removing the i-th row and the j-th column from A.

Ex

$$A = \begin{array}{cc} & \begin{array}{ccc} & j=2 & \end{array} \\ & \begin{pmatrix} 1 & \cancel{2} & -7 \\ 2 & \cancel{1} & 0 \\ \cancel{-3} & \cancel{6} & \cancel{1} \end{pmatrix} & i=3 \end{array}$$

$$A_{\widehat{32}} = \begin{pmatrix} 1 & -7 \\ 2 & 0 \end{pmatrix}$$

Lemma (Expansion of the determinant)

Let $A \in M_n(F)$.

(i) Expansion with respect to the j-th column:

pick $1 \leq j \leq n$, then:

(*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{\widehat{ij}} \qquad \leftarrow$$

(ii) Expansion with respect to the i-th row:

pick $1 \leq i \leq n$, then:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \det A_{\widehat{ij}} \qquad \leftarrow$$

$\Rightarrow$ powerful tool to compute determinants

Example

$$A = \begin{pmatrix} \overset{+}{1} & \overset{-}{2} & \overset{+}{-1} \\ \overset{-}{3} & \overset{+}{-1} & \overset{-}{1} \\ \overset{+}{4} & \overset{-}{2} & \overset{+}{-7} \end{pmatrix}$$

↑ 2nd Column

$$\det A = -(2)\begin{vmatrix} 3 & 1 \\ 4 & -7 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 4 & -7 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$

proof Expansion with respect to the j-th Column (row expansion formula follows by taking ^{T}).

Pick $1 \le j \le n$.

. $A = \left(A^{(1)} \middle| A^{(2)} \dots \middle| A^{(j)} \middle| \dots \middle| A^{(n)} \right)$

$A^{(j)} = \sum_{i=1}^{n} a_{ij} e_i$, $\quad A = (a_{ij})_{1 \le i,j \le n}$

. $\det A =$

$$\det\left(A^{(1)}, \dots, \sum_{i=1}^{n} a_{ij} e_i, \dots, A^{(n)} \right)$$

$$= \sum_{i=1}^{n} a_{ij} \det \underbrace{\left(A^{(1)}, \dots, \overset{\downarrow \, j\text{th}}{e_i}, \dots, A^{(n)} \right)}_{} \nearrow$$

$$\longleftarrow$$

$$\det\left(\begin{array}{c|c|c|c|c} & & j & & \\ A^{(1)} & \dots & \begin{array}{c} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \begin{array}{c} \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} & \dots & A^{(n)} \end{array} \right)$$

$$= (-1)^{\boxed{j-1}} \det \begin{pmatrix} 0 & & & & & \\ \vdots & & & & & \\ 1 & A^{(1)} & \cdots & A^{(j-1)} & A^{(j+1)} & \cdots & A^{(n)} \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix} \begin{matrix} \overset{1}{\uparrow} \\ \\ \leftarrow i \\ \\ \\ \\ \end{matrix}$$

$$= (-1)^{i-1} (-1)^{j-1} \det \begin{pmatrix} \boxed{1} & a_{i1} & \cdots & a_{ij-1} & a_{ij+1} & \cdots & a_{in} \\ 0 & & & & & & \\ | & & & \widehat{A_{ij}} \in \mathcal{M}_{n-1,n-1} & & & \\ 0 & & & & & & \end{pmatrix}$$

$$= (-1)^{i+j} \det\left(\widehat{A_{ij}}\right)$$

$\uparrow$

bloc

determinant formula

We have proved:

$$\det\left(A^{(1)}, \ldots, A^{(j-1)}, \overset{\overset{j}{\downarrow}}{e_i}, A^{(j+1)}, \ldots, A^{(n)}\right)$$

$$= (-1)^{i+j} \det\left(A_{\hat{i}\hat{j}}\right).$$

Now: $$\det A = \sum_{i=1}^{n} a_{ij} \det\left(A^{(1)}, \ldots, A^{(j-1)}, e_i, \ldots, A^{(n)}\right)$$

$$= \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det A_{\hat{i}\hat{j}} \qquad \square$$

Def (Adjugate matrix)

Let $A \in M_n(F)$. The adjugate matrix $\operatorname{adj}(A)$ is the $n \times n$ matrix with (i,j) entry given by:

$$(-1)^{i+j}\det\left(A_{\hat{ji}}\right)$$

$$\equiv$$

$$\left[\det\left(A^{(1)},\dots,A^{(j-1)},e_i,A^{(j)},\dots,A^{(n)}\right)\right]$$

$$\|$$

$$\left(\operatorname{adj}(A)\right)_{ji}$$

Thm Let $A \in M_n(F)$, then:

$$\operatorname{adj}(A)\,A = (\det A)\, I_d = \begin{pmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{pmatrix}$$

In particular, when A is invertible,

$$\boxed{A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)},$$

proof We just proved: $(*)$

- $$\det A = \sum_{i=1}^{n} (-1)^{i+j} \left(\det A_{\widehat{ij}}\right) a_{ij}$$

$$\underset{\substack{\uparrow \\ \text{def of} \\ \operatorname{adj}(A)}}{=} \sum_{i=1}^{n} \left(\operatorname{adj}(A)\right)_{ji} a_{ij} = \left(\operatorname{adj}(A)\, A\right)_{jj}$$

- For $j \neq k$, we have:

$$0 = \det\Big(A^{(1)}, \dots, \underset{\substack{\uparrow \\ j\text{-th column} \\ \text{vector}}}{A^{(k)}}, \dots, A^{(k)}, \dots, A^{(n)}\Big)$$

$$= \det\Big(A^{(1)}, \dots, \sum_{i=1}^{n} a_{ik} e_i, \dots, A^{(k)}, \dots, A^{(n)}\Big)$$

$$= \sum_{i=1}^{n} a_{ik} \left[\underbrace{\det\left(A^{(1)}, \dots, \underset{\underset{j}{\uparrow}}{e_i}, \dots, A^{(n)}\right)}_{\overset{\|}{\left(\operatorname{adj}(A)\right)_{ji}}} \right] \Bigg\}$$

$$= \sum_{i=1}^{n} \left(\operatorname{adj}(A)\right)_{ji} a_{ik}$$

$$= \left(\operatorname{adj}(A)\, A\right)_{jk} = 0 \quad \text{for } j \neq k.$$

Cramer rule

Prop Let $A \in M_n(F)$ be invertible. Let $b \in F^n$. Then the unique

Solution to $Ax=b$ is given by:

$$x_i = \frac{1}{\det A} \det\left(A_{\hat{i}}b\right), \quad 1 \leq i \leq n, \quad (*)$$

where: $A_{\hat{i}}b$ is obtained by replacing the i_th column of A by b.

⇝ Algorithmically, this avoids computing A^{-1}

proof A invertible, $\exists! x \in F^n \,/\, Ax=b$.

Let x be this solution, then:

$$\det\left(A_{\hat{i}}b\right) = \det\left(A^{(1)}, \dots, A^{(i-1)}, b, A^{(i+1)}, \dots, A^{(n)}\right)$$

$$= \det\left(A^{(1)}, \dots, A^{(i-1)}, Ax, A^{(i+1)}, \dots, A^{(n)}\right)$$

$$= \det\left(A^{(1)}, \dots, A^{(i-1)}, \sum_{j=1}^{n} x_j \cdot A^{(j)}, A^{(i+1)}, \dots, A^{(n)}\right)$$

↑
alternate
linear

$$= x_i \det\left(A^{(1)}, \dots, A^{(i-1)}, A^{(i)}, \dots, A^{(n)}\right)$$

$$= x_i \det A$$

$$\Rightarrow \quad x_i = \frac{\det A_{\widehat{ib}}}{\det A}, \qquad \square.$$