Lecture 12 Some properties of determinants.

$$\boxed{\textbf{Lemma} \quad A, B \in M_n(F), \text{ then: } \quad \det(AB) = (\det A)(\det B)}$$

proof let $d_A : F^n \times \dots \times F^n \longrightarrow F$

$$d_A(v_1, \dots, v_n) = \det(Av_1, \dots, Av_n)$$

- d_A is multilinear ($v_i \mapsto Av_i$ linear)
- d_A is alternating ($v_i = v_j \Rightarrow Av_i = Av_j$)

$\Rightarrow d_A$ is a volume form

$\Rightarrow d_A(Be_1, \dots, Be_n) = \det B \, \underline{d_A(e_1, \dots, e_n)}$

multilinearity (exercise)

$$= \det B \det A \underbrace{\det(e_1, \dots, e_n)}_{=1}$$

$\exists!$ up to a constant volume form

We have proved that: $d_A(Be_1, \dots, Be_n) = (\det B)(\det A)$.

On the other hand, by definition:

$$d_A(v_1, \dots, v_n) = \det(Av_1, \dots, Av_n)$$

$$\Rightarrow d_A(Be_1, \dots, Be_n) = \det(ABe_1, \dots, ABe_n)$$

$$= \det(AB) \underbrace{\det(e_1, \dots, e_n)}_{=1}$$

multilinearity + alternate

<u>Conclusion</u> $d_A(Be_1, \dots, Be_n) = (\det A)(\det B) =$

$\det(AB)$ $\square$.

Def $A \in M_n(F)$ we say that:

(i) A is singular if $\det A = 0$

(ii) A is non singular if $\det A \neq 0$.

Lemma A is invertible $\Rightarrow$ A is non singular.

proof A is invertible $\exists A^{-1}$:

$$AA^{-1} = A^{-1}A = I_d$$

$$\Rightarrow \det(AA^{-1}) = (\det A)(\det A^{-1}) = 1$$

$$\Rightarrow \det A \neq 0$$

$\square$.

Remark We have proved that:

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Thm Let $A \in M_n(\mathbb{F})$. TFAE:

(i) A is invertible

(ii) A is non singular

(iii) $r(A) = n$.

proof (i) $\Leftrightarrow$ (iii) done (rank nullity Th)

(i) $\Rightarrow$ (ii) (lemma above)

We need to show that (ii) $\Rightarrow$ (iii). Assume $r(A) < n$. Let us show that $\det A = 0$.

$$r(A) < n \Rightarrow \dim \operatorname{Span}\{c_1, \dots, c_n\} < n$$

$$\left(A = (c_1, \dots, c_n) \right)$$

$$\Rightarrow \exists (\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0) \; / \; \sum_{i=1}^{n} \lambda_i c_i = 0.$$

$$j \; / \; \lambda_j \neq 0, \quad c_j = -\frac{1}{\lambda_j} \sum_{i \neq j} \lambda_i c_i.$$

$$\Rightarrow \det A = \det (c_1, \dots, c_j, \dots, c_n)$$

$$= \det \left(c_1, \dots, -\frac{1}{\lambda_j} \sum_{i \neq j} \lambda_i c_i, \dots, c_n \right)$$

$$= 0 \qquad \Rightarrow \det A = 0$$

$\square$.

develop by multilinearity + alternate property

Rk $\rightsquigarrow$ Giving the sharp criterion for invertibility of a set of n linear equations with n unknowns:

$$\begin{cases} Y \in F^n \\ A \in \mathcal{M}_n(F) \end{cases}$$

$$\begin{cases} AX = Y \\ X \in F^n \end{cases} \iff \det A \neq 0 .$$

Determinant of linear maps

> **Lemma** Conjugate matrices have the same determinant

proof $\det(P^{-1}AP) = (\det P^{-1}) \det A (\det P)$

$$= \det A \frac{\det P}{\det P} = \det A \qquad \square$$

Def $\alpha : V \to V$ linear. We define

$$\det \alpha = \det [\alpha]_B$$

B any basis of V

→ this number does not depend on the choice of basis B.

Thm $\det : L(V,V) \to F$

satisfies:

(i) $\det \mathrm{Id} = 1$

(ii) $\det(\alpha\beta) = (\det \alpha)(\det \beta)$

$\alpha \circ \beta$

(iii) $\det \alpha \neq 0 \iff \alpha$ is invertible,

and in this case: $\det(\alpha^{-1}) = (\det \alpha)^{-1}$.

proof Reformulation of previous theorem for: $[x]_B$ $\rightsquigarrow$ nothing depends on the choice of B.

Determinant of block-triangular matrices

Lemma $A \in M_k(F)$, $B \in M_\ell(F)$, $C \in M_{k,\ell}(F)$

Let: $$M = \begin{array}{c c} & \begin{array}{cc} k & \ell \end{array} \\ \begin{array}{c} k \\ \ell \end{array} & \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \end{array} \in M_n(F) \quad (n = k+\ell)$$

Then: $$\det M = (\det A)(\det B).$$

proof $n = k + l$

$$M = \begin{array}{c|cc} & k & l \\ \hline k & A & C \\ l & 0 & B \end{array} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \left(m_{ij}\right)_{1 \le i,j \le n}$$

I need to compute:

$$\det M = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} m_{\sigma(i)i} \quad (*)$$

. Observe that:

$$m_{\sigma(i)i} = 0 \quad \text{if} \quad \begin{cases} i \le k \\ \sigma(i) > k \end{cases}$$

line   Column

. In $(*)$, we need only sum over $\sigma \in S_n$ s.th:

(i) $\forall j \in [\![1, k]\!],\quad \sigma(j) \in [\![1, k]\!]$

(ii) $\forall j \in [\![k+1, n]\!],\quad \sigma(j) \in [\![k+1, n]\!]$

In other words, we may restrict the summation $(*)$ to σ of the form:

$$\boxed{\begin{cases} \sigma_1 : \{1, \dots, k\} \longrightarrow \{1, \dots, k\}, \equiv \sigma_{|\{1,\dots,k\}} \\ \sigma_2 : \{k+1, \dots, n\} \longrightarrow \{k+1, \dots, n\} \equiv \sigma_{|\{k+1,\dots,n\}} \end{cases}}$$

(i) $\begin{cases} m_{\sigma(j)j} & = a_{\sigma(j)j} = a_{\sigma_1(j)j} \\ 1 \leq j \leq k, & \sigma(j) \in \{1, \dots, k\} \end{cases}$

(ii) $m_{\sigma(j),j} = b_{\sigma(j)j} = b_{\sigma_2(j)j}$

$k+1 \leq j \leq n, \quad \sigma(j) \in \{k+1, \dots, n\}$

. Observe that:

$$\sigma : \overbrace{\underbrace{1 \quad 2 \quad \cdots \quad k}_{\sigma_1 \downarrow}}^{k} \ \overbrace{\underbrace{k+1 \quad \cdots \quad n}_{\downarrow \sigma_2}}^{\ell}$$

$$(1 \ 2 \ \cdots \ k) \ (k+1 \ \cdots \ n)$$

$$\varepsilon(\sigma) = \varepsilon(\sigma_1)\,\varepsilon(\sigma_2)$$

S_k = bijections of $\{1, \ldots, k\} \to \{1, \ldots, k\}$

S_ℓ = ——— of $\{k+1, \ldots, n\} \to \{k+1, \ldots, n\}$

$\{1, \ldots, \ell\} \to \{1, \ldots, \ell\}$

$$\det M = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} m_{\sigma(i)\, i}$$

$$= \sum_{\substack{\sigma_1 \in S_k \\ \sigma_2 \in S_\ell}} \varepsilon(\sigma_1)\varepsilon(\sigma_2) \prod_{i=1}^{k} a_{\sigma_1(i)\, i} \prod_{j=k+1}^{n} b_{\sigma_2(j)\, j}$$

$$= (\det A)(\det B)$$

Cor $A_1, \dots, A_\ell$ are square matrices, then:

$$\det \begin{pmatrix} A_1 & & * \\ & A_2 & \\ 0 & & \ddots \quad A_\ell \end{pmatrix} = (\det A_1) \cdots (\det A_\ell)$$

proof By induction on the number of blocks.

In particular $A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $\lambda_i \in F$

$$\Rightarrow \det A = \prod_{i=1}^{n} \lambda_i$$

(you can also prove this directly using the det A formula)

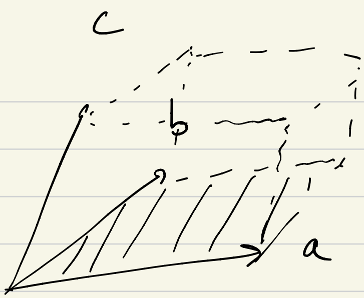
⚠ In general:

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \neq \det A \det D - \det B \det C$$

Remark Volume form?

$$\boxed{\begin{aligned} \mathbb{R}^3 : \quad \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 &\longrightarrow \mathbb{F} \\ (a, b, c) &\longmapsto (a \wedge b) \cdot c \end{aligned}}$$ $\equiv$ volume form

$$a \wedge b = \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix} \wedge \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix} = \begin{vmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{vmatrix}$$

$=$ 

$$\boxed{\vec{a} \wedge \vec{b} \cdot \vec{c}} \equiv \text{signed volume of } (\vec{a}, \vec{b}, \vec{c})$$

Exercise $\det(a, b, c) = (\vec{a} \wedge \vec{b}) \cdot \vec{c}$