Lecture 10   Bilinear forms

Def U, V vector spaces over F. Then:

$$\varphi : U \times V \to F \quad \text{is a bilinear form}$$

if it is linear in both components:

$$\varphi(u, .) : V \to F \quad \in V^* \qquad (\forall u \in U)$$

$$\varphi(., v) : U \to F \quad \in U^* \qquad (\forall v \in V)$$

Ex (i) $V \times V^* \to F$

$$(v, \theta) \mapsto \theta(v)$$

(ii) Scalar product on $U = V = \mathbb{R}^n$:

$$\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

$$\left(x \begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix}, y \begin{vmatrix} y_1 \\ \vdots \\ y_n \end{vmatrix} \right) \mapsto \psi(x, y) = \sum_{i=1}^{n} x_i y_i$$

(iii) $U = V = \mathcal{C}([0;1], \mathbb{R})$

$$\varphi(f, g) = \int_0^1 f(t) g(t)\, dt$$

(= infinite dimensional scalar product)

Def $\mathcal{B} = (e_1, \dots, e_m)$ basis of U

$\mathcal{C} = (f_1, \dots, f_n)$ basis of V

$\varphi : U \times V \to F$ bilinear form.

The matrix of φ wrt $\mathcal{B}$ and $\mathcal{C}$ is:

$$[\varphi]_{\mathcal{B}\,\mathcal{C}} = \big(\underbrace{\varphi(e_i, f_j)}_{\in K} \big)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Lemma $$\varphi(u, v) = [u]_{\mathcal{B}}^T \, [\varphi]_{\mathcal{B},\mathcal{C}} \, [v]_{\mathcal{C}} \qquad (*)$$

→ link between the bilinear form and its matrix in a basis.

<u>proof</u> $u = \sum_{i=1}^{m} \lambda_i e_i \, , \; v = \sum_{j=1}^{n} \mu_j f_j$

Then by linearity:

$$\varphi(u,v) = \varphi\left(\sum_{i=1}^{m} \lambda_i e_i, \sum_{j=1}^{n} \mu_j f_j\right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j \boxed{\varphi(e_i, f_j)}$$

bilinearity

$=$ claimed formula. □

<u>Remark</u> $[\varphi]_{\mathcal{B},\mathcal{C}}$ is the unique matrix s.t. $(*)$ holds.

Notation $\varphi : U \times V \to F$ bilinear determines two linear maps :

$$\varphi_L : U \to V^* \qquad \left(\begin{array}{l} \varphi_L(u) : V \to F \\ \qquad v \mapsto \varphi(u,v) \end{array} \right)$$

$$\varphi_R : V \to U^* \qquad \left(\begin{array}{l} \varphi_R(v) : U \to F \\ \qquad u \mapsto \varphi(u,v) \end{array} \right)$$

In particular : $\forall (u,v) \in U \times V$,

$$\varphi_L(u)(v) = \varphi(u,v) = \varphi_R(v)(u)$$

Lemma $\mathcal{B} = (e_1, \dots, e_m)$ of U

$\mathcal{B}^* = (\varepsilon_1, \dots, \varepsilon_m)$ of U^* dual

$\mathcal{C} = (f_1, \dots, f_n)$ of V

$\mathcal{C}^* = (\eta_1, \dots, \eta_n)$ of V^* dual

If : $A = [\varphi]_{\underline{B}, \mathcal{C}}$, then :

$$[\varphi_R]_{\mathcal{C}, \underline{B}^*} = A$$

$$[\varphi_L]_{\underline{B}, \mathcal{C}^*} = A^T$$

proof. $\varphi_L(e_i)(f_j) = \varphi(e_i, f_j) = A_{ij}$

$$\Rightarrow \varphi_L(e_i) = \sum A_{ij} \eta_j$$

$\cdot\ \varphi_R(f_j)(e_i) = \varphi(e_i, f_j) = A_{ij}$

$$\Rightarrow \varphi_R(f_j) = \sum A_{ij} \, e_i$$

□

Def

$\ker \varphi_L$: left kernel of φ

$\ker \varphi_R$: right kernel of φ

We say that φ is **non degenerate** if:

$$\ker \varphi_L = \{0\} \quad \text{and} \quad \ker \varphi_R = \{0\}.$$

Otherwise, we say that φ is degenerate.

Lemma $\mathcal{B}$ basis of U (U, V finite dimensional)

$\mathcal{C}$ basis of V

$\varphi : U \times V \longrightarrow F$ bilinear

$A = [\varphi]_{\mathcal{B}, \mathcal{C}}$

Then: φ non degenerate $\iff$ A is invertible.

Cor φ non degenerate $\Rightarrow$ $\dim U = \dim V$.

proof φ non degenerate

$\iff$ $\mathrm{Ker}\,\varphi_L = \{0\}$ and $\mathrm{Ker}\,\varphi_R = \{0\}$ $\iff$ $\begin{cases} n(A^T) = 0 \\ n(A) = 0 \end{cases}$

$\iff$ $r(A^T) = \dim U$, $r(A) = \dim V$ $\iff$ A invertible

↑ rank nullity Theorem

(and forces $\dim U = \dim V$) $\square$

Rk $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ non degenerate.

$(x, y) \mapsto \sum_{i=1}^{n} x_i y_i$

Cor When U and V are finite dimensional then choosing a non degenerate bilinear form $\varphi : U \times V \to F$ is equivalent to choosing an isomorphism $\varphi_L : U \to V^*$.

Def. $\cdot\ T \subset U$ (subset), we define:

$$T^\perp = \{ v \in V \,/\, \varphi(t, v) = 0,\ \forall t \in T \}$$

$\cdot\ S' \subset V$ (subset)

$$^\perp S' = \{ u \in U \,/\, \varphi(u, s) = 0\ \forall s \in S' \}$$

$\equiv$ orthogonal of resp T and S'

Prop $\mathcal{B}, \mathcal{B}'$ basis of U, $P = [\mathrm{Id}]_{\mathcal{B}', \mathcal{B}}$

$\mathcal{C}, \mathcal{C}'$ basis of V, $Q = [\mathrm{Id}]_{\mathcal{C}', \mathcal{C}}$

Let $\varphi : U \times V \to F$ bilinear form, then:

$$\boxed{[\varphi]_{\mathcal{B}',\mathcal{C}'} = P^T [\varphi]_{\mathcal{B},\mathcal{C}} \, Q .}$$

change of basis formula
for bilinear forms

proof

$$\varphi(u,v) = [u]_{\mathcal{B}}^T [\varphi]_{\mathcal{B},\mathcal{C}} [v]_{\mathcal{C}}$$

$$= \left(P [u]_{\mathcal{B}'} \right)^T [\varphi]_{\mathcal{B},\mathcal{C}} \left(Q [v]_{\mathcal{C}'} \right)$$

$$= [u]_{\mathcal{B}'}^T \underbrace{\left(P^T [\varphi]_{\mathcal{B},\mathcal{C}} \, Q \right)}_{[\varphi]_{\mathcal{B}',\mathcal{C}'}} [v]_{\mathcal{C}'} \qquad \square$$

Def-Lemma The rank of φ ($r(\varphi)$) is the rank of any matrix representing φ

$\rightsquigarrow$ Indeed: $r(P^T A Q) = r(A)$ for any invertible P, Q.

Remark $r(\varphi) = r(\varphi_R) = r(\varphi_L)$

(We computed matrices in a basis and $r(A) = r(A^T)$)

$\rightsquigarrow$ More applications later: scalar product.

Determinant and traces

Trace

Def Let $A \in M_n(F)$ $(= M_{n,n}(F))$ square $n \times n$ matrix. We define the trace of A

as : $\boxed{\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}}$

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \\ A_{n1} & & A_{nn} \end{pmatrix}$$

<u>Remark</u> $M_n(F) \longrightarrow F$ linear form.

$A \longmapsto \operatorname{Tr} A$

<u>Lemma</u> $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$

$\forall A, B \in M_n(F) \times M_n(F)$

<u>proof</u> $\operatorname{Tr}(AB) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} b_{ji} \right)$

$$= \sum_{j=1}^{n} \underbrace{\sum_{i=1}^{n} b_{ji} a_{ij}}_{(BA)_{jj}} = \operatorname{Tr}(BA)$$

$\square$

Cor Similar matrices have the same trace.

proof $\mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}(AP^{-1}P)$
$= \mathrm{Tr}(A)$ $\square$.

Def If $\alpha : V \to V$ linear, we can define :
$\mathrm{Tr}\,\alpha = \mathrm{Tr}([\alpha]_B)$ in any basis B (does not depend on the choice of the basis)

Lemma $\alpha : V \to V$ linear
$\alpha^* : V^* \to V^*$ dual map
Then : $\mathrm{Tr}\,\alpha = \mathrm{Tr}\,\alpha^*$

proof $\mathrm{Tr}\,\alpha = \mathrm{Tr}[\alpha]_B \underset{\uparrow}{=} \mathrm{Tr}\left([\alpha]_B^T\right)$

check

$$= \operatorname{Tr}\left([\alpha^*]_{\beta^*} \right) = \operatorname{Tr} \alpha^* \qquad \square$$