Lecture 1 Vector spaces, subspaces

Let F be an arbitrary field ($F = \mathbb{R}$ or $\mathbb{C}$)

Def (F vector space) An F-vector space (a vector space over F) is an abelian group $(V, +)$ equipped with a function

$$F \times V \longrightarrow V$$
$$(\lambda, v) \longmapsto \lambda v$$

VECTOR

SCALAR VECTORS

such that:

- $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$
- $(\lambda_1 + \lambda_2) v = \lambda_1 v + \lambda_2 v$
- $\lambda(\mu v) = (\lambda\mu) v$
- $1 . v = v$

→ We know how to:

- sum two vectors
- multiply a vector $v \in V$ by a scalar $\lambda \in F$.

Examples

(i) $n \in \mathbb{N}$, F^n: column vectors of length n with entries in F

$$v \in F^n \quad , \quad v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad , \quad x_i \in F \,, \; 1 \leq i \leq n .$$

$$\cdot \; v + w = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

$$\cdot \; \lambda v = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix} \quad , \quad \lambda \in F .$$

F^n is an F-vector space.

(ii). Any set X,

$$\mathbb{R}^X = \{ f : X \to \mathbb{R} \}$$

(set of real valued functions on X)

$\mathbb{R}^X$ $\mathbb{R}$ vector space:

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(\lambda f)(x) = \lambda f(x)$, $\lambda \in \mathbb{R}$

(iii) $M_{n,m}(F) \equiv n \times m$ F-valued matrices

$$\begin{array}{cc} & m \\ n & \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix} \end{array}$$

$a_{ij} \in F$

$1 \leq i \leq n$

$1 \leq j \leq m$

$\leadsto$ Check as an exercise that the above examples are vector spaces.

Rk The axiom of scalar multiplication imply that: $\forall v \in V, \ 0.v = 0$

Def (*Subspace*) Let V be a vector space over F. The subset U of V is a vector subspace of V (noted $U \leq V$) if:

- $\boxed{0 \in U}$
- $(u_1, u_2) \in U \times U \Rightarrow u_1 + u_2 \in U$
- $(\lambda, u) \in F \times U \Rightarrow \lambda u \in U$

$\Leftrightarrow$ $\forall (\lambda_1, \lambda_2) \in F \times F, \forall (v_1, v_2) \in U \times U,$
$\lambda_1 v_1 + \lambda_2 v_2 \in U$.

$\Rightarrow$ This property means that U is stable by :

- scalar multiplication
- vector addition

Exercise V $\mathbb{F}$ vector subspace

$U \leq V$

$\Rightarrow U$ $\mathbb{F}$ vector subspace.

Examples

① $V = \mathbb{R}^{\mathbb{R}}$ space of functions $\mathbb{R} \to \mathbb{R}$

$\mathcal{C}(\mathbb{R}) \leq V$ ——— $\mathcal{C}^0$ functions $\mathbb{R} \to \mathbb{R}$

$\mathbb{P}(\mathbb{R}) \leq \mathcal{C}(\mathbb{R})$ space of polynomials

② $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 , \; x_1 + x_2 + x_3 = t \right\}$

Check that this is a subspace for $t = 0$ only.

Prop Let V be an F vector space. Let $U, W \leq V$. Then: $U \cap W \leq V$.

proof. $\left.\begin{array}{l} 0 \in U \\ 0 \in W \end{array}\right| \Rightarrow 0 \in U \cap W$

. Stability. $(\lambda_1, \lambda_2) \in F^2$

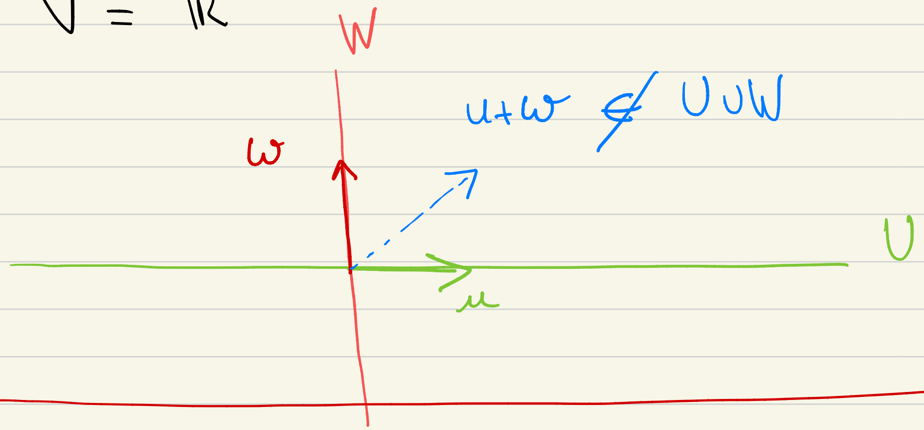
$(v_1, v_2) \in (U \cap W)^2$

$$\left.\begin{array}{l} \lambda_1 \underbrace{v_1}_{\in U} + \lambda_2 \underbrace{v_2}_{\in U} \in U \\ \lambda_1 \underbrace{v_1}_{\in U} + \lambda_2 \underbrace{v_2}_{\in U} \in W \end{array}\right| \Rightarrow \lambda_1 v_1 + \lambda_2 v_2 \in U \cap W$$

⚠ The union of two subspaces is generically <u>NOT</u> a subspace

$\rightsquigarrow$ it is typically not stable by addition

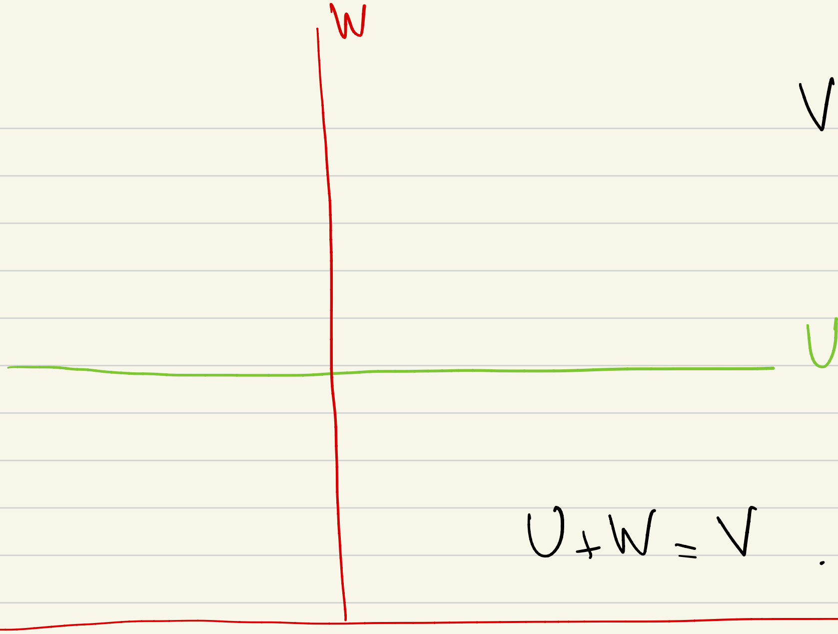
<u>Ex</u> $V = \mathbb{R}^2$



Def (Sum of subspaces) Let V be an $\mathbb{F}$ vector space. Let $U \leq V$, $W \leq V$.
The sum of U and W is the set:

$$U + W = \{ u + w \, , \, (u, w) \in U \times W \}$$

Ex. $V = \mathbb{R}^2$



$U + W = V$.

$\boxed{\text{Prop}}$ V F vector space

$(U \leq V,\ W \leq V) \Rightarrow U + W \leq V$.

Proof
$\cdot\ 0 = \underbrace{0}_{\in U} + \underbrace{0}_{\in W} \in U + W$

$\cdot\ \lambda_1 f + \lambda_2 g,\quad f \in U + W,\ g \in U + W$

$f = \underbrace{f_1}_{\in U} + \underbrace{f_2}_{\in W},\quad g = \underbrace{g_1}_{\in U} + \underbrace{g_2}_{\in W}$

$$\Rightarrow \lambda_1 f + \lambda_2 g$$

$$= \lambda_1 (f_1 + f_2) + \lambda_2 (g_1 + g_2)$$

$$= \underbrace{(\lambda_1 \underbrace{f_1}_{\in U} + \lambda_2 \underbrace{g_1}_{\in U})}_{\Downarrow\, U \leq V \atop \in U} + \underbrace{(\lambda_1 \underbrace{f_2}_{\in W} + \lambda_2 \underbrace{g_2}_{\in W})}_{\Downarrow\, W \leq V \atop \in V}$$

$$\in U + W.$$

Exercise: Show that $U+W$ is the smallest subspace of V which contains U and W.

. Subspaces and quotient.

Def (Quotient) Let V be an F vector space, let $U \leq V$. The quotient space V/U is the abelian group V/U equipped with the scalar multiplication:

$$\begin{aligned} F \times V/U &\longrightarrow V/U \\ (\lambda, v+U) &\longmapsto \lambda v + U \end{aligned} \qquad (*)$$

Prop V/U is an F vector space.

<u>Chem</u> that the multiplication operation $(*)$ is well defined. Indeed, $v_1 + U = v_2 + U$ then:

$\Rightarrow v_1 - v_2 \in U$

$\Rightarrow \quad \lambda(v_1 - v_2) \in U$

$\uparrow$

$U \leq V$ $\quad \Rightarrow \quad \lambda v_1 + U = \lambda v_2 + U \in V/U$

$\Rightarrow$ $(*)$ is well defined.

Exercise: Check that V/U is an F vector space.

Next lecture $\rightsquigarrow$ independance of vectors.