Michaelmas Term 2013 J. M. E. Hyland

Linear Algebra: Example Sheet 3

The first 10 questions cover the course as I see it and should ensure good understanding. The remainder are a mixed bag dealing with a number of mostly minor points. I hope some will prove instructive.

1. Show that none of the following matrices are conjugate:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Is the matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
conjugate to any of them? If so, which?

2. Find a basis with respect to which the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ has Jordan normal form. Hence compute the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n$.

3. Let V be a vector space of dimension n and α an endomorphism of V with $\alpha^n = 0$ but $\alpha^{n-1} \neq 0$. Show that there is a vector $\mathbf{x}$ such that $\mathbf{x}$, $\alpha(\mathbf{x})$, $\alpha^2(\mathbf{x})$, ... , $\alpha^{n-1}(\mathbf{x})$ is a basis for V.
(i) Let $p(t) = a_0 + a_1 t + a_2 t^2 .... + a_k t^k$ be a polynomial. What is the matrix for $p(\alpha)$ with respect to the basis given above?
(ii) Suppose that β is an endomorphism of V which commutes with α. Show that $\beta = p(\alpha)$ for some polynomial $p(t)$.
(iii) What can you deduce using (i) and (ii)?

4. Let A be a non-singular square matrix in Jordan normal form. What is the inverse of A? What is the Jordan normal form of the inverse of A?

5. (i) Show that the Jordan normal form of a 3×3 complex matrix is determined by its characteristic and minimal polynomials. Give an example to show that this fails for 4×4 matrices.
(ii) Let A be a complex 5×5 matrix with $A^4 = A^2 \neq A$. What are the possible minimum and characteristic polynomials? What are the possible Jordan normal forms?

6. Let P_2 be the space of polynomials in x, y of degree ≤ 2 in each variable. (So $\dim P_2 = 9$.)
Consider the map $D : P_2 \to P_2$ given by
$$D(f) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.$$
(i) What are the eigenvalues of the endomorphism D? Find the eigenspaces.
(ii) Determine the Jordan normal form of the endomorphism D.
(iii) Make a guess about what happens for the n^2-dimensional space P_n space of polynomials in x, y of degree $\leq n$ in each variable. (Prove it?)

7. Which of the following symmetric matrices are congruent to the identity matrix (a) over $\mathbb{C}$, (b) over $\mathbb{R}$ and (c) over $\mathbb{Q}$? (Try to get away with the minimum calculation.)
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

8. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.

$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

If B is the matrix of the form then there exists non-singular Q with $Q^t BQ$ diagonal with entries ± 1. Find such a Q in some representative cases.

9. (i) Show that the map $A \mapsto \mathrm{tr}(A.A^t)$ is a positive definite quadratic form on $\mathrm{Mat}_n(\mathbb{R})$, the space of $n \times n$ matrices.
(ii) Show that the map $A \mapsto \mathrm{tr}(A^2)$ is a quadratic form also on the space $\mathrm{Mat}_n(\mathbb{R})$. What is its rank and signature?

10. (i) Show that the quadratic form $2(x^2 + y^2 + z^2 + xy + yz + zx)$ is positive definite.
(ii) *Write down* an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$.
(iii) Compute the basis obtained by applying the Gram-Schmidt process to the standard basis.

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11. (i) Show that if $\alpha : V \to V$ is an endomorphism of a finite dimensional complex vector space, then there is a basis of V with respect to which the matrix of α is upper triangular. [*This is standard bookwork.*]
(ii) Let α and β be endomorphisms of a finite dimensional complex vector space V. Suppose that $\alpha\beta = \beta\alpha$. Does it follow that there is a basis of V with respect to which the matrices of both α and β are upper triangular?

12. Find the left and right kernels of the bilinear form with matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

with respect to the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$. Let $V = \langle \mathbf{e}_2, \mathbf{e}_3 \rangle$. Find $V^\perp$ and ${}^\perp V$. Give a basis $\mathbf{f}_1, \ldots, \mathbf{f}_4$ with respect to which the bilinear form has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

13. Suppose that ψ is a bilinear form on V. Take $U \leq V$ with $U = W^\perp$ some $W \leq V$. Suppose that $\psi|_U$ is non-singular. Show that ψ is also non-singular.

14. Suppose that $\psi : U \times V \to F$ is a bilinear form on U, V finite dimensional vector spaces. Show that there exist bases $\mathbf{e}_1, \ldots, \mathbf{e}_m$ for U and $\mathbf{f}_1, \ldots, \mathbf{f}_n$ for V such that when $\mathbf{x} = \sum_1^m x_i \mathbf{e}_i$ and $\mathbf{y} = \sum_1^n y_j \mathbf{f}_j$ then we have $\psi(\mathbf{x}, \mathbf{y}) = \sum_1^r x_k y_k$, where r is the rank of ψ. What are the dimensions of the left and right kernels of ψ?

15. Find the rank and signature of the form on $\mathbb{R}^n$ with matrix

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}.$$

16. Suppose that Q is a non-singular quadratic form on V of dimension $2m$. Suppose that Q vanishes on $U \leq V$ with $\dim U = m$. What is the signature of Q? Establish the following.
(i) There is a basis with respect to which Q has the form $x_1x_2 + x_3x_4 + \cdots x_{2m-1}x_{2m}$.
(ii) We can write $V = U \oplus W$ with q also vanishing on W.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.