Linear Algebra: Example Sheet 4 of 4

The first ten questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be harder; they are intended to be attempted only after completion of the first part.

1. The square matrices A and B over the field F are congruent if $B = P^T AP$ for some invertible matrix P over F. Which of the following symmetric matrices are congruent to the identity matrix over $\mathbb{R}$, and which over $\mathbb{C}$? (Which, if any, over $\mathbb{Q}$?) Try to get away with the minimum calculation.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.

$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

If A is the matrix of the first of these (say), find a non-singular matrix P such that $P^T AP$ is diagonal with entries ± 1.

3. (i) Show that the function $\psi(A, B) = \mathrm{tr}(AB^T)$ is a symmetric positive definite bilinear form on the space $\mathrm{Mat}_n(\mathbb{R})$ of all $n \times n$ real matrices. Deduce that $|\mathrm{tr}(AB^T)| \leq \mathrm{tr}(AA^T)^{1/2}\mathrm{tr}(BB^T)^{1/2}$.
(ii) Show that the map $A \mapsto \mathrm{tr}(A^2)$ is a quadratic form on $\mathrm{Mat}_n(\mathbb{R})$. Find its rank and signature.

4. Let $\psi : V \times V \to \mathbb{C}$ be a Hermitian form on a complex vector space V.
(i) Show that if $n > 2$ then $\psi(u, v) = \frac{1}{n} \sum_{k=1}^{n} \zeta^k \psi(u + \zeta^k v, u + \zeta^k v)$ where $\zeta = e^{2\pi i/n}$.
(ii) Find the rank and signature of ψ in the case $V = \mathbb{C}^3$ and

$$\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.$$

5. Show that the quadratic form $2(x^2 + y^2 + z^2 + xy + yz + zx)$ is positive definite. Compute the basis of $\mathbb{R}^3$ obtained by applying the Gram-Schmidt process to the standard basis.

6. Let $W \leq V$ with V an inner product space. An endomorphism π of V is called an *idempotent* if $\pi^2 = \pi$. Show that the orthogonal projection onto W is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.

7. Let S be an $n \times n$ real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.

8. An endomorphism α of a finite dimensional inner product space V is *positive definite* if it is self-adjoint and satisfies $\langle \alpha(\mathbf{x}), \mathbf{x} \rangle > 0$ for all non-zero $\mathbf{x} \in V$.
(i) Prove that a positive definite endomorphism has a unique positive definite square root.
(ii) Let α be an invertible endomorphism of V and α^* its adjoint. By considering $\alpha^*\alpha$, show that α can be factored as $\beta\gamma$ with β unitary and γ positive definite.

9. Let V be a finite dimensional complex inner product space, and let α be an endomorphism on V. Assume that α is *normal*, that is, α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that α and α^* have a common eigenvector $\mathbf{v}$, and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle \mathbf{v} \rangle^\perp$ is invariant under both α and α^*. Deduce that there is an orthonormal basis of eigenvectors of α.

10. Find a linear transformation which reduces the pair of real quadratic forms

$$2x^2 + 3y^2 + 3z^2 - 2yz, \qquad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$

to the forms

$$X^2 + Y^2 + Z^2, \qquad \lambda X^2 + \mu Y^2 + \nu Z^2$$

for some $\lambda, \mu, \nu \in \mathbb{R}$ (which should turn out in this example to be integers).

Does there exist a linear transformation which reduces the pair of real quadratic forms $x^2 - y^2$, $2xy$ simultaneously to diagonal forms?

---

11. Let $f_1, \cdots, f_t, f_{t+1}, \cdots, f_{t+u}$ be linear functionals on the finite dimensional real vector space V. Show that $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \cdots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \cdots - f_{t+u}(\mathbf{x})^2$ is a quadratic form on V. Suppose Q has rank $p+q$ and signature $p-q$. Show that $p \leq t$ and $q \leq u$.

12. Suppose that Q is a non-degenerate quadratic form on V of dimension $2m$. Suppose that Q vanishes on $U \leq V$ with $\dim U = m$. What is the signature of Q? Establish the following.
(i) There is a basis with respect to which Q has the form $x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m}$.
(ii) We can write $V = U \oplus W$ with Q also vanishing on W.

13. Suppose that α is an orthogonal endomorphism on the finite-dimensional real inner product space V. Prove that V can be decomposed into a direct sum of mutually orthogonal α-invariant subspaces of dimension 1 or 2. Determine the possible matrices of α with respect to orthonormal bases in the cases where V has dimension 1 or dimension 2.

14. Show that if A is an $m \times n$ real matrix of rank n then A^TA is invertible. Is there a corresponding result for complex matrices?

15. Prove Hadamard's Inequality: if A is a real $n \times n$ matrix with $|a_{ij}| \leq k$, then

$$|\det A| \leq k^n n^{n/2} .$$

16. Let P_n be the ($n+1$-dimensional) space of real polynomials of degree $\leq n$. Define

$$\langle f, g \rangle = \int_{-1}^{+1} f(t)g(t)dt .$$

Show that $\langle \ , \ \rangle$ is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by

$$\alpha(f)(t) = (1-t^2)f''(t) - 2tf'(t)$$

is self-adjoint. What are the eigenvalues of α?
Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1-t^2)^k$. Prove the following.
(i) For $i \neq j$, $\langle s_i, s_j \rangle = 0$.
(ii) $s_0, \ldots, s_n$ forms a basis for P_n.
(iii) For all $1 \leq k \leq n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α. (Give its eigenvalue.)
What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? (Calculate the first few terms?)

17. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots + a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

18. Let A be a $2n \times 2n$ alternating matrix over a field F. Show that the determinant of A is a square. In fact $\det(A) = \mathrm{pf}(A)^2$ where $\mathrm{pf}(A)$ is a homogeneous polynomial of degree n in the entries of A (called the *Pfaffian* of A). Assuming this fact, show that every matrix in the *symplectic group*

$$\mathrm{Sp}_{2n}(F) = \{P \in \mathrm{GL}_{2n}(F) \mid P^TJP = J\}, \qquad \text{where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

has determinant $+1$.