Linear Algebra: Non-degenerate Bilinear Forms

These notes cover some material related to the linear algebra course, marginally beyond that specified in the schedules. This includes the classification of skew-symmetric bilinear forms (recall that symmetric bilinear forms were covered in lectures). The last section of the course is on inner products, i.e. positive definite symmetric bilinear forms (case $F = \mathbb{R}$), respectively positive definite Hermitian forms (case $F = \mathbb{C}$). In these notes we generalise some of the results of that section to non-degenerate forms.

1. Non-degeneracy

Let $V$ and $W$ be finite dimensional vector spaces over a field $F$. Recall that $V^* = \text{L}(V,F)$ is the dual space of $V$. If $\psi : V \times W \to F$ is a bilinear form then there are linear maps

$$
\psi_L : V \to W^*; \quad v \mapsto (w \mapsto \psi(v,w))
$$

$$
\psi_R : W \to V^*; \quad w \mapsto (v \mapsto \psi(v,w)).
$$

Linearity of $\psi$ in the second argument shows that $\psi_L$ is linear, and hence an element of $W^*$, whereas linearity of $\psi$ in the first argument shows that $\psi_L$ itself is linear. (The same comments apply to $\psi_R$ with obvious modifications.)

**Theorem 1.1.** Any two of the following statements implies the third.

(i) $\text{Ker}(\psi_L) = \{0\}$, i.e. $\psi(v,w) = 0$ for all $w \in W$ implies $v = 0$.

(ii) $\text{Ker}(\psi_R) = \{0\}$, i.e. $\psi(v,w) = 0$ for all $v \in V$ implies $w = 0$.

(iii) $\dim V = \dim W$.

**Proof:** Statement (i) shows that $\dim V \leq \dim W^* = \dim W$, and likewise (ii) shows that $\dim W \leq \dim V^* = \dim V$. So (i) and (ii) imply $\dim V = \dim W$.

Now suppose that (i) and (iii) hold. Then $\psi_L : V \to W^*$ is an isomorphism. Pick a basis $v_1, \ldots, v_n$ for $V$. Then $\psi_L(v_1), \ldots, \psi_L(v_n)$ is a basis for $W^*$. Let $w_1, \ldots, w_n$ be the dual basis for $W$. Then $\psi(v_i, w_j) = \psi_L(v_i)(w_j) = \delta_{ij}$. If $w \in \text{Ker}(\psi_R)$, say $w = \sum \lambda_j w_j$ for some $\lambda_j \in F$, then $\lambda_i = \psi(v_i, w) = \psi_R(w)(v_i) = 0$ for all $i$. Hence $w = 0$ and this proves (ii). The deduction of (iii) from (i) and (ii) is similar. \qed

**Definition 1.2.** A bilinear form $\psi : V \times W \to F$ is **non-degenerate** if it satisfies the conditions of Theorem 1.1.

Equivalently, $\psi$ is non-degenerate if and only if $\text{rank}(\psi) = \dim V = \dim W$. Recall that the rank of $\psi$ is the rank of any matrix representing it.

**Remark 1.3.** The set of bilinear forms $V \times V \to F$ under pointwise operations form a vector space over $F$. We may identify this space with $\text{L}(V, V^*)$ via $\psi \mapsto \psi_L$. The non-degenerate bilinear forms correspond to the isomorphisms from $V$ to $V^*$. 

2. **Orthogonal complements**

Let $\psi : V \times V \to F$ be a bilinear form. We assume that either $\psi$ is symmetric, i.e. $\psi(u, v) = \psi(v, u)$ for all $u, v \in V$, or $\psi$ is skew-symmetric, i.e. $\psi(u, v) = -\psi(v, u)$ for all $u, v \in V$. Then for $W \leq V$ we define the orthogonal subspace

$$W^\perp = \{ v \in V : \psi(v, w) = 0 \text{ for all } w \in W \}.$$ 

The restriction of $\psi$ to $W$ (denoted $\psi|_W$ although we really mean $\psi|_{W \times W}$) is non-degenerate if and only if $W \cap W^\perp = \{0\}$.

**Remark 2.1.** It is possible for the restriction of a non-degenerate form to be degenerate. For example let $\psi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by $(u, v) \mapsto u^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v$, and let $W = \langle e_1 \rangle$. Then $\psi$ is non-degenerate but $\psi|_W$ is degenerate. This is in contrast to the situation for positive definite forms: the restriction of a positive definite form is always positive definite.

**Theorem 2.2.** Let $\psi : V \times V \to F$ be bilinear and either symmetric or skew-symmetric. Let $W \leq V$ be a subspace. Then $\dim(W) + \dim(W^\perp) = \dim(V) + \dim(W \cap V^\perp)$.

**Proof:** Let $\alpha$ be the restriction of $\psi$ to $W$, i.e.

$$\alpha : W \to V^* \quad w \mapsto (v \mapsto \psi(v, w)).$$

The rank-nullity theorem says

$$\dim(W) = \dim \ker(\alpha) + \dim \im(\alpha).$$

But $\ker(\alpha) = \{ w \in W : \psi(v, w) = 0 \text{ for all } v \in V \} = W \cap V^\perp$ and

$$\im(\alpha)^0 = \{ v \in V : \theta(v) = 0 \text{ for all } \theta \in \im(\alpha) \} = \{ v \in V : \alpha(w)(v) = 0 \text{ for all } w \in W \} = \{ v \in V : \psi(v, w) = 0 \text{ for all } w \in W \} = W^\perp.$$

Since for $U \leq V$ we have $\dim(U) + \dim(U^\circ) = \dim V$ it follows that

$$\dim(W) = \dim(W \cap V^\perp) + (\dim(V) - \dim(W^\perp)).$$

□

**Corollary 2.3.** Let $\psi : V \times V \to F$ be bilinear and either symmetric or skew-symmetric. Let $W \leq V$ be a subspace. Then

$\psi|_W$ is non-degenerate $\iff V = W \oplus W^\perp$.

**Proof:** “$\Rightarrow$” Since $\psi|_W$ is non-degenerate we have $W \cap W^\perp = \{0\}$. Therefore $W + W^\perp$ is a direct sum. Then $\dim(W \oplus W^\perp) = \dim(W) + \dim(W^\perp) \geq \dim(V)$ by Theorem 2.2. Hence $V = W \oplus W^\perp$.

“$\Leftarrow$” If $V = W \oplus W^\perp$ then $W \cap W^\perp = \{0\}$ and $\psi|_W$ is non-degenerate. □

We used a special case of Corollary 2.3 (with $W$ a 1-dimensional subspace) in the proof that a symmetric bilinear form can be diagonalised.
3. Skew-symmetric forms

We assume that $F$ is a field of characteristic not 2.

**Definition 3.1.** A bilinear form $\psi : V \times V \to F$ is alternating if $\psi(v, v) = 0$ for all $v \in V$.

**Lemma 3.2.** Let $\psi : V \times V \to F$ be a bilinear form. Then $\psi$ is alternating if and only if it is skew-symmetric.

**Proof:** $\Rightarrow$ By bilinearity

\[
\psi(u + v, u + v) = \psi(u, u) + \psi(u, v) + \psi(v, u) + \psi(v, v).
\]

Since $\psi$ is alternating this reduces to $\psi(u, v) = -\psi(v, u)$, i.e. $\psi$ is skew-symmetric.

$\Leftarrow$ Since $\psi(v, v) = -\psi(v, v)$ (and $2 \neq 0$ in $F$) we get $\psi(v, v) = 0$. \[\square\]

**Theorem 3.3.** Let $V$ be a finite dimensional vector space over $F$ and let $\psi : V \times V \to F$ be an alternating bilinear form. Then there exists a basis $B$ for $V$ such that

\[
[\psi]_B = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

In particular the rank of $\psi$ is even.

**Proof:** The proof is by induction on $\dim V$. If $\psi$ is the zero form then we are done. Otherwise pick $v_1, v_2 \in V$ with $\psi(v_1, v_2) \neq 0$. Then $\psi$ alternating implies $v_1$ and $v_2$ are linearly independent. Replacing $v_2$ by $cv_2$ for some non-zero $c \in F$ we may assume that $\psi(v_1, v_2) = 1$. Put $W = \langle v_1, v_2 \rangle$. Then $\psi|_W$ has matrix \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). By Corollary 2.3 (or a direct argument of the sort we used in the proof for symmetric forms) we have $V = W \oplus W^\perp$. Applying the induction hypothesis to $\psi|_{W^\perp}$ gives a basis $v_3, \ldots, v_n$ for $W^\perp$. Then $v_1, \ldots, v_n$ is the required basis for $V$. \[\square\]

**Corollary 3.4.** If a finite dimensional vector space $V$ admits a non-degenerate alternating bilinear form then $\dim V$ is even.

4. Adjoints

Although we will meet adjoints in the section of the course on inner products, they can be defined more generally for non-degenerate bilinear forms. As before $V$ will be a finite dimensional vector space over $F$.

**Lemma 4.1.** Let $\phi$ and $\psi$ be bilinear forms on $V$ with $\psi$ non-degenerate. Then there exists a unique $\alpha \in \text{End}(V)$ such that

\[
\phi(v, w) = \psi(v, \alpha(w))
\]

for all $v, w \in V$. 

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**Note:** The above text is a natural representation of the given document, formatted in plain text as requested. The document contains mathematical content typical of a mathematical textbook or lecture notes. The text is structured into sections, definitions, lemmas, and theorems, with proofs and examples provided where necessary. The document covers topics such as skew-symmetric forms, alternating forms, and the definition of adjoints. The mathematical notation and terminology are consistent with standard usage in mathematics, particularly in the context of linear algebra and related fields.
Proof: There are linear maps
\[ \phi_R : V \to V^* ; \quad w \mapsto (v \mapsto \phi(v, w)) \]
\[ \psi_R : V \to V^* ; \quad w \mapsto (v \mapsto \psi(v, w)). \]

Since \( \psi \) is non-degenerate, \( \psi_R \) is an isomorphism. We put \( \alpha = \psi_R^{-1} \circ \phi_R \). Then
\[ \psi_R \circ \alpha = \phi_R \]
\[ \implies \psi_R(\alpha(w))(v) = \phi_R(w)(v) \quad \text{for all } v, w \in V \]
\[ \implies \psi(v, \alpha(w)) = \phi(v, w) \quad \text{for all } v, w \in V. \]

Uniqueness: Suppose \( \alpha_1, \alpha_2 \in \text{End}(V) \) are solutions. Then \( \psi(v, \alpha_1(w)) = \phi(v, w) = \psi(v, \alpha_2(w)) \) for all \( v, w \in V \). Then \( \psi(v, \alpha_1(w) - \alpha_2(w)) = 0 \) for all \( v, w \in V \), and by non-degeneracy of \( \psi \) it follows that \( \alpha_1 = \alpha_2 \). \( \Box \)

Theorem 4.2. Let \( \psi : V \times V \to F \) be a non-degenerate bilinear form. For each \( \alpha \in \text{End}(V) \) there exists a unique \( \alpha^* \in \text{End}(V) \) such that
\[ \psi(\alpha(v), w) = \psi(v, \alpha^*(w)) \]
for all \( v, w \in V \). We call \( \alpha^* \) the adjoint of \( \alpha \).

Proof: Define \( \phi : V \times V \to F \) by \( (v, w) \mapsto \psi(\alpha(v), w) \). Then \( \phi \) is bilinear and Lemma 4.1 constructs \( \alpha^* \). \( \Box \)

Remark 4.3. If \( \psi \) is non-degenerate then \( \psi_R : V \to V^* \) is an isomorphism. If we identify \( V \) and \( V^* \) via this map then the adjoint \( \alpha^* \) works out as being the same as the dual map (as defined in the section on dual spaces, and also denoted \( \alpha^* \).)