Linear Algebra: Example Sheet 1 of 4

The first twelve questions cover the relevant part of the course and should ensure a good understanding. The remaining questions may or may not be harder; they should only be attempted after completion of the first part.

1. Let $\mathbb{R}^{\mathbb{R}}$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $\mathbb{R}^{\mathbb{R}}$?
(a) The set C of continuous functions.
(b) The set $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0,1]\}$.
(c) The set $\{f \in C : f(t) \to 0 \text{ as } t \to \infty\}$.
(d) The set $\{f \in C : f(t) \to 1 \text{ as } t \to \infty\}$.
(e) The set of solutions of the differential equation $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = 0$.
(f) The set of solutions of $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = \sin t$.
(g) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.
(h) The set of solutions of $(\ddot{x}(t))^4 + (x(t))^2 = 0$.

2. Suppose that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a basis for V. Which of the following are also bases?
(a) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n$;
(b) $\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1$;
(c) $\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n - \mathbf{e}_1$;
(d) $\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n \mathbf{e}_1$.

3. Show that a vector space V is finite dimensional if and only if every linearly independent subset $S \subset V$ is finite. Deduce that a subspace of a finite dimensional vector space is always finite dimensional. (You may quote the Steinitz Exchange Lemma, but should otherwise work from first principles.)

4. Let T, U and W be subspaces of V.
(i) Show that $T \cup U$ is a subspace of V only if either $T \leq U$ or $U \leq T$.
(ii) Give explicit counter-examples to the following statements:

$$\text{(a)} \quad T + (U \cap W) = (T + U) \cap (T + W); \qquad \text{(b)} \quad (T + U) \cap W = (T \cap W) + (U \cap W).$$

(iii) Show that each of the equalities in (ii) can be replaced by a valid inclusion of one side in the other.
(iv) Show that if $T \leq W$, then $(T + U) \cap W = T + (U \cap W)$.

5. Let $\mathbb{R}^{\mathbb{N}}$ be the space of all real sequences $(x_n)_{n \geq 1}$. Which of the following rules define a linear map $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$? In each case where the answer is 'yes' describe the kernel and image of both T and T^2.
(a) $y_n = x_n + 1$.
(b) $y_n = x_{n+1} - x_n$.
(c) $y_n = x_{n+1} x_n + x_{n+2}$.
(d) $y_n = x_1$ if n is odd and $y_n = x_2$ if n is even.
(e) $y_n = x_{n-1}$ if $n \geq 2$ and $y_1 = 0$.
(f) $y_n = \sum_{i=1}^{n} x_i$.

6. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. (Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a, b]$ denotes the space of continuous functions defined on the closed interval $[a, b]$.)
(a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
(b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
(c) $V = C[0, 1]$, $W = C[-1, 1]$.
(d) $V = C[0, 1]$, $W = \{f \in C[0, 1] : f(0) = 0, f \text{ continuously differentiable }\}$.
(e) $V = \mathbb{R}^2$, $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
(f) $V = \mathbb{R}^4$, $W = C[0, 1]$.
(g) (Harder:) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

7. Let U be a vector subspace of $\mathbb{R}^n$. Show that there is a finite subset I of $\{1, 2, \ldots, n\}$ for which the subspace $W = \langle \{\mathbf{e}_i : i \in I\} \rangle$ is a complementary subspace to U in $\mathbb{R}^n$.

8. (Revision from IA.) Let

$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ \ 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ \ x_2 = x_3 = x_4\}.$$

Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}\,.$$

9. Let $\alpha : U \to V$ be a linear map between two finite dimensional vector spaces and let W be a vector subspace of U. Show that the restriction of α to W is a linear map $\alpha|_W : W \to V$ which satisfies

$$\mathrm{r}(\alpha) \geq \mathrm{r}(\alpha|_W) \geq \mathrm{r}(\alpha) - \dim(U) + \dim(W)\ .$$

Give examples (with $W \neq U$) to show that either of the two inequalities can be an equality.

10. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.
Write down bases for the domain and range with respect to which the matrix of α is the identity.

11. Let $U_1, \ldots, U_k$ be subspaces of a vector space V and let B_i be a basis for U_i. Show that the following statements are equivalent:

    (i) $U = \sum_i U_i$ is a direct sum, *i.e.* every element of U can be written uniquely as $\sum_i u_i$ with $u_i \in U_i$.

    (ii) $U_j \cap \sum_{i \neq j} U_i = \{0\}$ for all j.

    (iii) The B_i are pairwise disjoint and their union is a basis for $\sum_i U_i$.

    Given an example where $U_i \cap U_j = \{0\}$ for all $i \neq j$, yet $U_1 + \ldots + U_k$ is not a direct sum.

12. Let Y and Z be subspaces of the finite dimensional vector spaces V and W, respectively. Show that $R = \{\alpha \in L(V, W) : \alpha(\mathbf{x}) \in Z \text{ for all } \mathbf{x} \in Y\}$ is a subspace of the space $L(V, W)$ of all linear maps from V to W. What is the dimension of R?

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13. X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample. [Look at $\mathbb{R}^3$.]

14. Let U be a proper subspace of the finite-dimensional vector space V. Find a basis for V containing no element of U.

15. (Another version of the Steinitz Exchange Lemma.) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ be linearly independent subsets of a vector space V, and suppose $r \leq s$. Show that it is possible to choose distinct indices $i_1, i_2, \ldots, i_r$ from $\{1, 2, \ldots, s\}$ such that, if we delete each $\mathbf{y}_{i_j}$ from Y and replace it by $\mathbf{x}_j$, the resulting set is still linearly independent. Deduce that any two maximal linearly independent subsets of a finite-dimensional vector space have the same size.

16. Let T, U, V, W be vector spaces over the same field and let $\alpha : T \to U, \beta : V \to W$ be fixed linear maps. Show that the mapping $\Phi : L(U, V) \to L(T, W)$ which sends θ to $\beta \circ \theta \circ \alpha$ is linear. If the spaces are finite-dimensional and α and β have rank r and s respectively, find the rank of Φ.

17. Let $\mathbb{F}_p$ be the field of integers modulo p, where p is a prime number. Let V be a vector space of dimension n over $\mathbb{F}_p$. How many vectors are there in V? How many bases? How many automorphisms does V have? How many k-dimensional subspaces are there in V?