Michaelmas Term 2006 J. Saxl

Linear Algebra: Example Sheet 2

The first twelve questions cover the relevant part of the course and should ensure good understanding. The remaining questions may or may not be harder; they are intended to be attempted only after completion of the first part. Questions 2, 6 and 7 are largely for revision.

1. (i) Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V. Show that $V \geq \mathrm{Im}(\alpha) \geq \mathrm{Im}(\alpha^2) \geq \dots$ and $\{\mathbf{0}\} \leq N(\alpha) \leq N(\alpha^2) \leq \dots$ If $r_k = \mathrm{r}(\alpha^k)$, deduce that $r_k \geq r_{k+1}$. Show also that $r_k - r_{k+1} \geq r_{k+1} - r_{k+2}$ for $k \geq 0$. [Consider the restriction of α to $\mathrm{Im}(\alpha^k)$.] Deduce that if, for some $k \geq 0$, we have the equality $r_k = r_{k+1}$, then $r_k = r_{k+\ell}$ for all $\ell \geq 0$.
   (ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $\mathrm{r}(\alpha)$ and $\mathrm{r}(\alpha^2)$?

2. For what values of a and b does the system of simultaneous linear equations
$$\begin{aligned} x + y + z &= 1 \\ ax + 2y + z &= b \\ a^2x + 4y + z &= b^2 \end{aligned}$$
have (i) a unique solution, (ii) no solution, (iii) many solutions?

3. Let $\lambda \in F$. Evaluate the determinant of the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. [Note that the sum of all columns of A has all entries equal.]

4. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are conjugate when regarded as matrices over $\mathbb{C}$, then they are conjugate as matrices over $\mathbb{R}$.

5. (i) Let V be a non-trivial vector space of finite dimension. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \iota$.
   (ii) Let V be the space of infinitely differentiable real functions. Find endomorphisms α, β of V which do satisfy $\alpha\beta - \beta\alpha = \iota$.

6. Compute the characteristic polynomials of the matrices
$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
Which of the matrices are diagonalizable over $\mathbb{C}$? Which over $\mathbb{R}$?

7. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$
The second and third matrices commute; find a basis with respect to which they are both diagonal.

8. Let V be a vector space, let $\pi_1, \pi_2, \dots, \pi_k$ be endomorphisms of V such that $\iota = \pi_1 + \dots + \pi_k$ and $\pi_i\pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \dots \oplus U_k$, where $U_j = \mathrm{Im}(\pi_j)$.
   Let α be an endomorphism on the vector space V, satisfying the equation $\alpha^3 = \alpha$. Prove directly that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

9. Let A be a square complex matrix of finite order - that is, $A^m = I$ for some m. Show that A can be diagonalized. [You can use a theorem.]

10. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way.
    Are the eigenspaces $\ker(\alpha - \lambda I)$ and $\ker(\alpha^2 - \lambda^2 I)$ necessarily the same?

11. (Another proof of the Diagonizability Theorem.) Let V be a vector space of finite dimension. If α_1 and α_2 are endomorphisms of V, show that the nullities satisfy the Sylvester law $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$. Deduce that if α is an endomorphism of V such that $p(\alpha) = 0$ for some polynomial $p(t)$ which is a product of distinct linear factors, then α is diagonizable.

12. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorizes as $A = BC$ with $B \in M_{m\times r}$ and $C \in M_{r\times n}$. Using the fact that $(BC)^t = C^t B^t$, deduce that the (column) rank of A^t equals r.

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13. Let A be an $n \times m$ matrix. Prove that if B is an $m \times n$ matrix then

$$\mathrm{r}(AB) \leq \min(\mathrm{r}(A), \mathrm{r}(B)).$$

At the start of each year the jovial and popular Dean of Muddling (pronounced Chumly) College organizes m parties for the n students of the College. Each student is invited to exactly k parties, and every two students are invited to exactly one party in common. Naturally $k \geq 2$. Let $P = (p_{ij})$ be the $n \times m$ matrix defined by

$$p_{ij} = \begin{cases} 1 & \text{if student i is invited to party j} \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the matrix PP^t and find its rank. Deduce that $m \geq n$.
(Fisher's inequality according to TWK.)
After the Master's cat has been found dyed green, maroon and purple on successive nights, the other fellows insist that next year $k = 1$. Why does the proof above now fail, and what will, in fact, happen next year? (The answer required is mathematical rather than sociological in nature.)

14. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A)\,, \quad (ii)\ \det(\mathrm{adj}A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}(\mathrm{adj}A) = (\det A)^{n-2}A\,.$$

What happens if A is singular?

Show that the rank of the matrix adj$A$ is $\mathrm{r}(\mathrm{adj}(A)) = \begin{cases} n & \text{if } \mathrm{r}(A) = n; \\ 1 & \text{if } \mathrm{r}(A) = n-1; \\ 0 & \text{if } \mathrm{r}(A) \leq n-2. \end{cases}$

15. Let $f(x) = a_0 + a_1x + \ldots + a_nx^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_n & a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \cdots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^n f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

16. Let $\alpha : V \to V$ be an endomorphism of a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

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