Michaelmas Term 2005 J. Saxl

Linear Algebra: Preliminaries

This is NOT one of the official example sheets of the course.

This sheet contains a few questions to revise some of the linear algebra covered in the IA Algebra and Geometry course last year. It is not one of the official example sheets - there will be four of these, the first one appearing soon. You may wish to revisit exercises 1 to 8 on Professor Körner's sheet 1 for Algebra and Geometry IA - this is still available on the web, if you lost your copy. A few other examples appear below. You may wish to look at these soon, to get some early practice. A page of notes for revision, compiled by a previous lecturer, Professor Hyland, appears on the website for the course, under the heading Recapitulation. This and further example sheets are also taken largely from ones prepared a couple of years ago by Professor Hyland.

1. Let U be the subset of $\mathbb{R}^3$ consisting of all vectors $\mathbf{x}$ satisfying the various conditions below. In which of these cases is U a vector space over $\mathbb{R}$?
(a) $x_1 > 0$.
(b) either $x_1 = 0$ or $x_2 = 0$.
(c) $x_1 + x_2 = 0$.
(d) $x_1 + x_2 = 1$.
(e) $x_1 + x_2 + x_3 = 0$ and $x_1 - x_3 = 0$.

2. For each of the vector spaces found in the question above, what is the dimension?

3. Show that the four vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$ form a linearly dependent set, but that any proper subset of them is linearly independent.

4. Which of the following are bases for $\mathbb{R}^3$?
$$\text{(a)} \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}; \quad \text{(b)} \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

5. Find the ranks of the following matrices A, and give bases for the kernel and image of the linear maps $\mathbf{x} \mapsto A\mathbf{x}$.
$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad ; \quad \begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix} \quad ; \quad \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}.$$

6. Find a basis with respect to which $\begin{pmatrix}0&-2\\1&3\end{pmatrix}$ is diagonal. Hence compute the nth power $\begin{pmatrix}0&-2\\1&3\end{pmatrix}^n$.

7. For some of the matrices A in question 5, calculate its characteristic polynomial χ_A and check that $\chi_A(A) = 0$.

8. For what values of a and b does the system of simultaneous linear equations
$$\begin{aligned} x + y + z &= 1 \\ ax + 2y + z &= b \\ a^2x + 4y + z &= b^2 \end{aligned}$$
have (i) a unique solution, (ii) no solution, (iii) many solutions? When solutions do exist, find them.

Comments, corrections and queries can be sent to me at saxl@dpmms.cam.ac.uk.