Michaelmas Term 2004 J. Saxl

Linear Algebra: Preliminaries

This is NOT one of the official example sheets of the course.

This sheet contains a few questions to revise some of the linear algebra covered in the IA Algebra and Geometry course last year. It is not one of the official example sheets - there will be four of these, the first one appearing soon. You may wish to revisit exercises 1 to 8 on Professor Körner's sheet 1 for Algebra and Geometry IA - this is still available on the web, if you lost your copy. A few other examples appear below. You may wish to look at these soon, to get some early practice. A page of notes for revision, compiled by last year's lecturer, Professor Hyland, appears on the web site for the course, under the heading Recapitulation. This and further example sheets are also taken largely from ones prepared last year by Professor Hyland.

1. Let U be the subset of $\mathbb{R}^3$ consisting of all vectors $\mathbf{x}$ satisfying the various conditions below. In which of these cases is U a vector space over $\mathbb{R}$?
(a) $x_1 > 0$.
(b) either $x_1 = 0$ or $x_2 = 0$.
(c) $x_1 + x_2 = 0$.
(d) $x_1 + x_2 = 1$.
(e) $x_1 + x_2 + x_3 = 0$ and $x_1 - x_3 = 0$.

2. For each of the vector spaces found in the question above, what is the dimension?

3. Show that the four vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$ form a linearly dependent set, but that any proper subset of them is linearly independent.

4. Which of the following are bases for $\mathbb{R}^3$?
$$\text{(a)}\ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}; \quad \text{(b)}\ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

5. Find the ranks of the following matrices A, and give bases for the kernel and image of the linear maps $\mathbf{x} \mapsto A\mathbf{x}$.
$$\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad ; \quad \begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix} \quad ; \quad \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}.$$

6. Find a basis with respect to which $\begin{pmatrix}0&-2\\1&3\end{pmatrix}$ is diagonal. Hence compute the nth power $\begin{pmatrix}0&-2\\1&3\end{pmatrix}^n$.

7. For some of the matrices A in question 5, calculate its characteristic polynomial χ_A and check that $\chi_A(A) = 0$.

8. For what values of a and b does the system of simultaneous linear equations
$$\begin{aligned} x + y + z &= 1 \\ ax + 2y + z &= b \\ a^2x + 4y + z &= b^2 \end{aligned}$$
have (i) a unique solution, (ii) no solution, (iii) many solutions? When solutions do exist, find them.

Comments, corrections and queries can be sent to me at `saxl@dpmms.cam.ac.uk`.