Groups Rings and Modules: Example Sheet 3 of 4

All rings in this course are commutative with a 1.

1. Show that $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\omega]$ are Euclidean domains, where $\omega = \frac{1}{2}(-1+\sqrt{-3})$. Show also that the usual Euclidean function $\phi(r) = N(r)$ does not make $\mathbb{Z}[\sqrt{-3}]$ into a Euclidean domain. Could there be some other Euclidean function ϕ making $\mathbb{Z}[\sqrt{-3}]$ into a Euclidean domain?

2. Show that the ideal $(2, 1+\sqrt{-7})$ in $\mathbb{Z}[\sqrt{-7}]$ is not principal.

3. Find an element of $\mathbb{Z}[\sqrt{-17}]$ that is a product of two irreducibles and also a product of three irreducibles.

4. Determine whether or not the following rings are fields, PIDs, UFDs, integral domains:
$$\mathbb{Z}[X],\quad \mathbb{Z}[X]/(X^2+1),\quad \mathbb{Z}[X]/(2, X^2+1),\quad \mathbb{Z}[X]/(2, X^2+X+1),\quad \mathbb{Z}[X]/(3, X^3-X+1).$$

5. Determine which of the following polynomials are irreducible in $\mathbb{Q}[X]$:
$$X^4+2X+2,\quad X^4+18X^2+24,\quad X^3-9,\quad X^3+X^2+X+1,\quad X^4+1,\quad X^4+4.$$

6. Let R be an integral domain. The *greatest common divisor* (gcd) of non-zero elements a and b in R is an element d in R such that d divides both a and b, and if c divides both a and b then c divides d.
   (i) Show that the gcd of a and b, if it exists, is unique up to multiplication by a unit.
   (ii) In lectures we have seen that, if R is a UFD, the gcd of two elements exists. Give an example to show that this is not always the case in an integral domain.
   (iii) Show that if R is a PID, the gcd of elements a and b exists and can be written as $ra + sb$ for some $r, s \in R$. Give an example to show that this is not always the case in a UFD.
   (iv) Explain briefly how, if R is a Euclidean domain, the Euclidean algorithm can be used to find the gcd of any two non-zero elements. Use the algorithm to find the gcd of $11+7i$ and $18-i$ in $\mathbb{Z}[i]$.

7. Find all ways of writing the following integers as sums of two squares: $221,\ 209\times 221,\ 121\times 221,\ 5\times 221$.

8. By considering factorisations in $\mathbb{Z}[\sqrt{-2}]$, show that the only integer solutions to the equation $x^2+2=y^3$ are $x=\pm 5,\ y=3$.

9. Let R be any ring. Show that the ring $R[X]$ is a principal ideal domain if and only if R is a field. Can every ideal in $\mathbb{C}[X,Y]$ be generated by two elements?

10. Exhibit an integral domain R and a (non-zero, non-unit) element of R that is not a product of irreducibles.

11. Let $\mathbb{F}_q$ be a finite field with q elements.

   (i) Show that the prime subfield K (that is, the smallest subfield) of $\mathbb{F}_q$ has p elements for some prime number p. Show that $\mathbb{F}_q$ is a vector space over K and deduce that $q = p^n$, for some n.

   (ii) Assuming that a field with p^n elements exists, show that $\mathrm{GL}_n(\mathbb{F}_p)$ contains an element of order $p^n - 1$.

Further Questions

12. (i) Consider the polynomial $f = X^3Y + X^2Y^2 + Y^3 - Y^2 - X - Y + 1$ in $\mathbb{C}[X, Y]$. Write it as an element of $(\mathbb{C}[X])[Y]$, that is collect together terms in powers of Y, and then use Eisenstein's criterion to show that f is prime in $\mathbb{C}[X, Y]$.

    (ii) Let F be any field. Show that the polynomial $f = X^2 + Y^2 - 1$ is irreducible in $F[X, Y]$, unless F has characteristic 2. What happens in that case?

13. Show that the subring $\mathbb{Z}[\sqrt{2}]$ of $\mathbb{R}$ is a Euclidean domain. Show that the units are $\pm(1 \pm \sqrt{2})^n$ for $n \geqslant 0$.

14. If a UFD has at least one irreducible, must it have infinitely many (pairwise non-associate) irreducibles?

15. Use your answer to Question 11 to show that if p and ℓ are primes, and ℓ is odd, then every Sylow ℓ-subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ is cyclic.

16. Let $\mathbb{F}_4 = \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1) = \{0, 1, \omega, \omega + 1\}$, a field with four elements.

    Show that the groups $\mathrm{SL}_2(\mathbb{F}_4)$ and $\mathrm{PSL}_2(\mathbb{F}_5)$ both have order 60. By exhibiting two Sylow 5-subgroups and using some questions from Example Sheet 1, or otherwise, show that they are both isomorphic to the alternating group A_5. Show that $\mathrm{SL}_2(\mathbb{F}_5)$ and $\mathrm{PGL}_2(\mathbb{F}_5)$ both have order 120, but that only one of these is isomorphic to S_5.

    [You may find it helpful to show, using the Cayley-Hamilton theorem or otherwise, that the order of an element $I \neq A \in \mathrm{SL}_2(\mathbb{F}_4)$ is uniquely determined by its trace.]