Groups Rings and Modules: Example Sheet 1 of 4

1. (i) What are the orders of elements of the group S_4? How many elements are there of each order?
   (ii) How many subgroups of order 2 are there in S_4? Of order 3? How many cyclic subgroups are there of order 4?
   (iii) Find a non-cyclic subgroup $V \leqslant S_4$ of order 4. How many such subgroups are there?
   (iv) Find a subgroup $D \leqslant S_4$ of order 8. How many such subgroups are there?
2. (i) Show that A_4 has no subgroups of index 2. Exhibit a subgroup of index 3.
   (ii) Show that A_5 has no subgroups of index 2, 3, or 4. Exhibit a subgroup of index 5.
   (iii) Show that A_5 is generated by $(12)(34)$ and (135).
3. Calculate the size of the conjugacy class of (123) as an element of S_4, as an element of S_5, and as an element of S_6. Find in each case its centraliser. Hence calculate the size of the conjugacy class of (123) in A_4, in A_5, and in A_6.
4. Suppose that $H, K \lhd G$ with $H \cap K = 1$. Show that any element of H commutes with any element of K. Hence show that $HK \cong H \times K$.
5. Let p be a prime number, and G be a non-abelian group of order p^3.
   (i) Show that the centre $Z(G)$ of G has order p.
   (ii) Show that if $g \notin Z(G)$ then its centraliser $C(g)$ has order p^2.
   (iii) Hence determine the sizes and numbers of conjugacy classes in G.
6. (i) For $p = 2, 3$ find a Sylow p-subgroup of S_4, and find its normaliser.
   (ii) For $p = 2, 3, 5$ find a Sylow p-subgroup of A_5, and find its normaliser.
7. Show that there are no simple groups of orders 441 or 351.
8. Let p, q, and r be prime numbers, not necessarily distinct. Show that no group of order pq is simple. Show that no group of order pq^2 is simple. Show that no group of order pqr is simple.
9. (i) Show that any group of order 15 is cyclic.
   (ii) Show that any group of order 30 has a normal subgroup of order 15.
10. (Semi-direct product) Let N and H be groups, and $\phi : H \to \mathrm{Aut}(N)$ a homomorphism. Show that we can define a group operation on the set $N \times H$ by
$$(n_1, h_1) \bullet (n_2, h_2) = (n_1 \cdot \phi(h_1)(n_2), h_1 \cdot h_2).$$
    Show that the resulting group G contains copies of N and H as subgroups, that N is normal in G, that $NH = G$, and that $N \cap H = 1$.

    By finding an element of order 3 in $\mathrm{Aut}(C_7)$, construct a non-abelian group of order 21.

Further Questions

11. Let p be a prime number. How many elements of order p are there in S_p? What are their centralisers? How many Sylow p-subgroups are there? What are the orders of their normalisers? If q is another prime number which divides $p-1$, show that there exists a non-abelian group of order pq.

12. Show that there are no simple groups of order 300 or 112.

13. Show that a group G of order 1001 contains normal subgroups of order 7, 11, and 13. Hence show that G is cyclic. [Hint: You may want to use Question 4.]

14. Let G be a simple group of order 60. Deduce that $G \cong A_5$, as follows. Show that G has six Sylow 5-subgroups. By considering the conjugation action of the set of Sylow 5-subgroups, show that G is isomorphic to a subgroup $G \leqslant A_6$ of index 6. By considering the action of A_6 on A_6/G, show that that there is an automorphism of A_6 taking G to A_5.

15. Let G be a group of order 60 which has more than one Sylow 5-subgroup. Show that G is simple.

16. Let G be a finite group with cyclic and non-trivial Sylow 2-subgroup. By considering the permutation representation of G on itself, show that G has a normal subgroup of index 2. [Hint: Show that a generator for the Sylow subgroup induces an odd permutation of G.]

17. (Frattini argument) Let $K \lhd G$ and P be a Sylow p-subgroup of K. Show that any element $g \in G$ may be written as $g = nk$ with $n \in N_G(P)$ and $k \in K$, and hence that $G = N_G(P)K$. [Hint: Observe that P and $g^{-1}Pg$ are both Sylow p-subgroups of K.] Deduce that $G/K \cong N_G(P)/N_K(P)$.