All rings in this course are commutative and have a multiplicative identity.

1. Let \( \omega = \frac{1}{2} (1 + \sqrt{-3}) \in \mathbb{C} \), let \( R = \{ a + b\omega : a, b \in \mathbb{Z} \} \), and let \( F = \{ a + b\omega : a, b \in \mathbb{Q} \} \). Show that \( R \) is a subring of \( \mathbb{C} \), and that \( F \) is a subfield of \( \mathbb{C} \). What are the units of \( R \)?

2. An element \( r \) of a ring \( R \) is called nilpotent if \( r^n = 0 \) for some \( n \).
   (i) What are the nilpotent elements of \( \mathbb{Z}/6\mathbb{Z} \)? Of \( \mathbb{Z}/8\mathbb{Z} \)? Of \( \mathbb{Z}/24\mathbb{Z} \)? Of \( \mathbb{Z}/1000\mathbb{Z} \)?
   (ii) Show that if \( r \) is nilpotent then \( r \) is not a unit, but \( 1 + r \) and \( 1 - r \) are units.
   (iii) Show that set of the nilpotent elements form an ideal \( N \) of \( R \). What are the nilpotent elements in the quotient ring \( R/N \)?

3. Let \( r \) be an element of a ring \( R \). Show that the polynomial \( 1 + rX \in R[X] \) is a unit if and only if \( r \) is nilpotent. Is it possible for the polynomial \( 1 + X \) to be a product of two non-units?

4. Show that if \( I \) and \( J \) are ideals in the ring \( R \), then so is \( I \cap J \), and the quotient \( R/(I \cap J) \) is isomorphic to a subring of the product \( R/I \times R/J \).

5. Let \( I_1 \subset I_2 \subset I_3 \subset \cdots \) be ideals in a ring \( R \). Show that the union \( I = \bigcup_{n=1}^{\infty} I_n \) is also an ideal. If each \( I_n \) is proper, explain why \( I \) must be proper.

6. Write down a prime ideal in \( \mathbb{Z} \times \mathbb{Z} \) that is not maximal. Explain why in a finite ring all prime ideals are maximal.

7. Explain why, for \( p \) a prime number, there is a unique ring of order \( p \). How many rings are there of order \( 4 \)?

8. Let \( R \) be an integral domain and \( F \) be its field of fractions. Suppose that \( \phi : R \to K \) is an injective ring homomorphism from \( R \) to a field \( K \). Show that \( \phi \) extends to an injective homomorphism \( \Phi : F \to K \) from \( F \) to \( K \). What happens if we do not assume that \( \phi \) is injective?

9. Let \( R \) be any ring. Show that the ring \( R[X] \) is a principal ideal domain if and only if \( R \) is a field.

10. An element \( r \) of a ring \( R \) is called idempotent if \( r^2 = r \).
    (i) What are the idempotent elements of \( \mathbb{Z}/6\mathbb{Z} \)? Of \( \mathbb{Z}/8\mathbb{Z} \)? Of \( \mathbb{Z}/24\mathbb{Z} \)? Of \( \mathbb{Z}/1000\mathbb{Z} \)?
    (ii) Show that if \( r \) is idempotent then so is \( r' = 1 - r \), and that \( rr' = 0 \). Show also that the ideal \( (r) \) is naturally a ring, and that \( R \) is isomorphic to \( (r) \times (r') \).

11. Let \( F \) be a field, and let \( R = F[X, Y] \) be the polynomial ring in two variables.
    (i) Let \( I \) be the principal ideal \( (X - Y) \) of \( R \). Show that \( R/I \cong F[X] \).
    (ii) Describe \( R/I \) when \( I = (X^2 + Y) \).
    (iii) What can you say about \( R/(X^2 - Y^2) \)? Is it an integral domain? Does it have nilpotent or idempotent elements? 
    (iv) Show that \( \mathbb{C}[X, Y]/(X^2 + Y^2 - 1) \cong \mathbb{C}[T, T^{-1}] \). [Hint: Think about trigonometric functions.]
Additional Questions

12. Is every abelian group the additive group of some ring?

13. Let $I$ be an ideal of the ring $R$ and $P_1, \ldots, P_n$ be prime ideals of $R$. Show that if $I \subset \bigcup_{i=1}^{n} P_i$, then $I \subset P_i$ for some $i$.

14. A sequence $\{a_n\}$ of rational numbers is a Cauchy sequence if $|a_n - a_m| \to 0$ as $m, n \to \infty$, and $\{a_n\}$ is a null sequence if $a_n \to 0$ as $n \to \infty$. Quoting any standard results from Analysis, show that the set of Cauchy sequences with componentwise addition and multiplication form a ring $C$, and that the null sequences form a maximal ideal $N$.

Deduce that $C/N$ is a field, with a subfield which may be identified with $\mathbb{Q}$. Explain briefly why the equation $x^2 = 2$ has a solution in this field.

15. Let $\wp$ be a set of prime numbers. Write $\mathbb{Z}_\wp$ for the collection of all rationals $m/n$ (in lowest terms) such that the only prime factors of the denominator $n$ are in $\wp$.

   (i) Show that $\mathbb{Z}_\wp$ is a subring of the field $\mathbb{Q}$ of rational numbers.
   (ii) Show that any subring $R$ of $\mathbb{Q}$ is of the form $\mathbb{Z}_\wp$ for some set $\wp$ of primes.
   (iii) Given (ii), what are the maximal subrings of $\mathbb{Q}$?

16. Show that there is no isomorphism as in Question 11 (iv) if both instances of $\mathbb{C}$ are replaced by $\mathbb{Q}$.

Comments or corrections to or257@cam.ac.uk