IB Groups, Rings, and Modules // Example Sheet 2

All rings in this course are commutative and have a multiplicative identity.

1. Let $\omega = \frac{1}{2}(1+\sqrt{-3}) \in \mathbb{C}$, let $R = \{a + b\omega : a, b \in \mathbb{Z}\}$, and let $F = \{a + b\omega : a, b \in \mathbb{Q}\}$. Show that R is a subring of $\mathbb{C}$, and that F is a subfield of $\mathbb{C}$. What are the units of R?

2. *An element r of a ring R is called nilpotent if $r^n = 0$ for some n.*
   (i) What are the nilpotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? Of $\mathbb{Z}/1000\mathbb{Z}$?
   (ii) Show that if r is nilpotent then r is not a unit, but $1 + r$ and $1 - r$ are units.
   (iii) Show that set of the nilpotent elements form an ideal N of R. What are the nilpotent elements in the quotient ring R/N?

3. Let r be an element of a ring R. Show that the polynomial $1 + rX \in R[X]$ is a unit if and only if r is nilpotent. Is it possible for the polynomial $1 + X$ to be a product of two non-units?

4. Show that if I and J are ideals in the ring R, then so is $I \cap J$, and the quotient $R/(I \cap J)$ is isomorphic to a subring of the product $R/I \times R/J$.

5. Let $I_1 \subset I_2 \subset I_3 \subset \cdots$ be ideals in a ring R. Show that the union $I = \bigcup_{n=1}^{\infty} I_n$ is also an ideal. If each I_n is proper, explain why I must be proper.

6. Write down a prime ideal in $\mathbb{Z} \times \mathbb{Z}$ that is not maximal. Explain why in a finite ring all prime ideals are maximal.

7. Explain why, for p a prime number, there is a unique ring of order p. How many rings are there of order 4?

8. Let R be an integral domain and F be its field of fractions. Suppose that $\phi : R \to K$ is an injective ring homomorphism from R to a field K. Show that ϕ extends to an injective homomorphism $\Phi : F \to K$ from F to K. What happens if we do not assume that ϕ is injective?

9. Let R be any ring. Show that the ring $R[X]$ is a principal ideal domain if and only if R is a field.

10. *An element r of a ring R is called idempotent if $r^2 = r$.*
    (i) What are the idempotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? Of $\mathbb{Z}/1000\mathbb{Z}$?
    (ii) Show that if r is idempotent then so is $r' = 1 - r$, and that $rr' = 0$. Show also that the ideal (r) is naturally a ring, and that R is isomorphic to $(r) \times (r')$.

11. Let F be a field, and let $R = F[X, Y]$ be the polynomial ring in two variables.
    (i) Let I be the principal ideal $(X - Y)$ of R. Show that $R/I \cong F[X]$.
    (ii) Describe R/I when $I = (X^2 + Y)$.
    (iii) What can you say about $R/(X^2 - Y^2)$? Is it an integral domain? Does it have nilpotent or idempotent elements? ...
    (iv) Show that $\mathbb{C}[X, Y]/(X^2 + Y^2 - 1) \cong \mathbb{C}[T, T^{-1}]$. [Hint: Think about trigonometric functions.]

Additional Questions

12. Is every abelian group the additive group of some ring?

13. Let I be an ideal of the ring R and $P_1, \ldots, P_n$ be prime ideals of R. Show that if $I \subset \bigcup_{i=1}^n P_i$, then $I \subset P_i$ for some i.

14. A sequence $\{a_n\}$ of rational numbers is a *Cauchy sequence* if $|a_n - a_m| \to 0$ as $m, n \to \infty$, and $\{a_n\}$ is a *null sequence* if $a_n \to 0$ as $n \to \infty$. Quoting any standard results from Analysis, show that the set of Cauchy sequences with componentwise addition and multiplication form a ring C, and that the null sequences form a maximal ideal N.

    Deduce that C/N is a field, with a subfield which may be identified with $\mathbb{Q}$. Explain briefly why the equation $x^2 = 2$ has a solution in this field.

15. Let ϖ be a set of prime numbers. Write $\mathbb{Z}_\varpi$ for the collection of all rationals m/n (in lowest terms) such that the only prime factors of the denominator n are in ϖ.

    (i) Show that $\mathbb{Z}_\varpi$ is a subring of the field $\mathbb{Q}$ of rational numbers.

    (ii) Show that any subring R of $\mathbb{Q}$ is of the form $\mathbb{Z}_\varpi$ for some set ϖ of primes.

    (iii) Given (ii), what are the maximal subrings of $\mathbb{Q}$?

16. Show that there is no isomorphism as in Question 11 (iv) if both instances of $\mathbb{C}$ are replaced by $\mathbb{Q}$.

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