

### Complex Analysis IB: 2025-26 – Sheet 3

1. Let  $g(z) = p(z)/q(z)$  be a rational function with  $\deg(q) \geq \deg(p) + 2$ . Show that the sum of the residues of  $g$  over all its poles is zero.
2. Let  $f$  be a meromorphic function on  $\mathbb{C}$  for which  $|f(z)| \rightarrow \infty$  as  $|z| \rightarrow \infty$ . Show that  $f$  cannot have poles at all integer points.
3. Evaluate the following integrals:

$$(a) \int_0^\pi \frac{d\theta}{4 + \sin^2 \theta}; \quad (b) \int_0^\infty \frac{x^2 dx}{(x^2 + 4)^2(x^2 + 9)};$$
$$(c) \int_0^\infty \sin x^2 dx; \quad (d) \int_0^\infty \frac{\ln(x^2 + 1)}{x^2 + 1} dx.$$

4. For  $-1 < \alpha < 1$  and  $\alpha \neq 0$ , compute

$$\int_0^\infty \frac{x^\alpha}{1 + x + x^2} dx.$$

5. Establish the following refinement of the Fundamental Theorem of Algebra. Let  $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$  be a polynomial of degree  $n$ , and let  $A = \max\{|a_i|, 0 \leq i \leq n-1\}$ . Then  $p(z)$  has  $n$  roots (counted with multiplicity) in the disk  $\{|z| < A + 1\}$ .
6. Let  $p(z) = z^5 + z$ . Find all  $z$  such that  $|z| = 1$  and  $\operatorname{Im} p(z) = 0$ . Calculate  $\operatorname{Re} p(z)$  for such  $z$ . Sketch the curve  $p \circ \gamma$ , where  $\gamma(t) = e^{2\pi it}$ , and hence determine the number of  $z$  (counted with multiplicity) such that  $|z| < 1$  and  $p(z) = x$  for each  $x \in \mathbb{R}$ .
7. (i) For a positive integer  $N$ , let  $\gamma_N$  be the square contour with vertices  $(\pm 1 \pm i)(N + 1/2)$ . Show that there exists  $C > 0$  such that for every  $N$ ,  $|\cot \pi z| < C$  on  $\gamma_N$ .  
(ii) By integrating  $\frac{\pi \cot \pi z}{z^2 + 1}$  around  $\gamma_N$ , show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = \frac{1 + \pi \coth \pi}{2}.$$

- (iii) Evaluate  $\sum_{n=0}^{\infty} (-1)^n / (n^2 + 1)$ .
8. (i) Show that  $z^4 + 12z + 1$  has exactly three zeros in the annulus  $\{1 < |z| < 4\}$ .  
(ii) Prove that  $z^5 + 2 + e^z$  has exactly three zeros in the half-plane  $\{z \mid \operatorname{Re}(z) < 0\}$ .
9. Show that the equation  $z \sin z = 1$  has only real solutions.

[Hint: Find the number of real roots in the interval  $[-(n + 1/2)\pi, (n + 1/2)\pi]$  and compare with the number of zeroes of  $z \sin z - 1$  in a square box  $\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)| < (n + 1/2)\pi\}$ .]

10. Let  $w \in \mathbb{C}$ , and let  $\gamma, \delta: [0, 1] \rightarrow \mathbb{C}$  be closed curves such that for all  $t \in [0, 1]$ ,  $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$ . By computing the winding number  $I(\sigma; 0)$  about the origin for the closed curve  $\sigma(t) = (\delta(t) - w)/(\gamma(t) - w)$ , show that  $I(\gamma; w) = I(\delta; w)$ .
- (ii) If  $w \in \mathbb{C}$ ,  $r > 0$ , and  $\gamma$  is a closed curve which does not meet  $B(w; r)$ , show that  $I(\gamma; w) = I(\gamma; z)$  for every  $z \in B(w; r)$ . Deduce that if  $\gamma$  is a closed curve in  $\mathbb{C}$  and  $U$  is the complement of  $\gamma$ , then the function  $w \mapsto I(\gamma; w)$  is a locally constant function on  $U$ .
11. Let  $U$  be a domain, let  $f : U \rightarrow \mathbb{C}$  be holomorphic and suppose  $a \in U$  with  $f'(a) \neq 0$ . Show that for  $r > 0$  sufficiently small,

$$g(w) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{zf'(z)}{f(z) - w} dz$$

defines a holomorphic function  $g$  in a neighbourhood of  $f(a)$  which is inverse to  $f$ .

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