Complex Analysis Example Sheet 1 **Lent 2024**

Comments on and/or corrections to the questions on this sheet are always welcome, and may be emailed to me at `hkrieger@dpmms.cam.ac.uk`.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a real linear map. Regarding T as a map from $\mathbb{C}$ into $\mathbb{C}$, show that there exist unique complex numbers A, B such that for every $z \in \mathbb{C}$, $T(z) = Az + B\overline{z}$. Show that T is complex differentiable if and only if $B = 0$.

2. (i) Let $f : U \to \mathbb{C}$ be a holomorphic function on a domain U. Show that f is constant if any one of its real part, modulus or argument is constant.

(ii) Find all entire functions of the form $f(x + iy) = u(x) + iv(y)$ where u and v are real valued.

(iii) Find all entire functions with real part $x^3 - 3xy^2$.

3. Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$ and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \qquad \text{for } z = x + iy \neq 0.$$

Show that f satisfies the Cauchy-Riemann equations at 0 but it not differentiable there.

4. (i) Denote by Log the principal branch of the logarithm. If $z \in \mathbb{C}$, show that $n\,\mathrm{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z \qquad \text{for all } z \in \mathbb{C}.$$

(ii) Defining $z^\alpha = \exp(\alpha\,\mathrm{Log}\,\mathrm{z})$ for $\alpha \in \mathbb{C}$ and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz} z^\alpha = \alpha z^{\alpha - 1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold?

5. (i) Verify directly that e^z, $\cos z$ and $\sin z$ satisfy the Cauchy-Riemann equations everywhere.

(ii) Find the set of complex numbers z for which $|e^{iz}| > 1$, and the set of those for which $|e^z| \leq e^{|z|}$.

(iii) Find the zeros of $1 + e^z$ and of $\cosh z$.

6. Prove that each of the following series converges uniformly on the corresponding subset of $\mathbb{C}$:

$$(a) \sum_{n=1}^{\infty} \sqrt{n} e^{-nz} \quad \text{on } \{\, z : 0 < r \leq \mathrm{Re}(z) \,\}; \qquad (b) \sum_{n=1}^{\infty} \frac{2^n}{z^n + z^{-n}} \quad \text{on } \{z : |z| \leq r < 1/2\}.$$

7. Find conformal equivalences between the following pairs of domains:

(i) the sector $\{z \in \mathbb{C} \mid -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc $D(0, 1)$;

(ii) the lens $\{z \in \mathbb{C} \mid |z - 1| < \sqrt{2} \text{ and } |z + 1| < \sqrt{2}\}$ and $D(0, 1)$;

(iii) the strip $S = \{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.

By considering a suitable bounded solution of Laplace's equation $u_{xx} + u_{yy} = 0$ on S, find a non-constant harmonic function on Q which is constant on its boundary axes.

8. (i) Show that the most general Möbius transformation which maps the unit disk onto itself has the form $z \mapsto \lambda \frac{z-a}{\bar{a}z-1}$, with $|a| < 1$ and $|\lambda| = 1$. [*Hint: first show that these maps form a group.*]

(ii) Find a Möbius transformation taking the region between the circles $\{|z| = 1\}$ and $\{|z-1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: a circle can be described by an equation of the shape* $|z-a|/|z-b| = \ell$.]

(iii) Find a conformal map from an infinite strip onto an annulus. Can such a map be the restriction to the strip of a Möbius map?

9. Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

10. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivatives) on the domains indicated:

$$(a)\ \frac{1}{z} - \frac{1}{z-1} \quad (0 < |z| < 1); \qquad (b)\ \frac{z}{1+z^2} \quad (1 < |z| < \infty).$$

11. Let $U \subset \mathbb{C}$ be a domain, and let $u : U \to \mathbb{R}$ be a C^2 harmonic function. Show that if $z_0 \in U$ then for any disk $D = D(z_0, r) \subset U$, there is a holomorphic function $f : D \to \mathbb{C}$ such that $u = \mathrm{Re}(f)$ on D. Show by an example that this need *not* hold globally; that is, there exists a choice of domain U and C^2 harmonic function u on U so that u is not the real part of any holomorphic function on U.