COMPLEX ANALYSIS EXAMPLES 3, LENT 2022

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1. The Weierstrass approximation theorem in real analysis says that every continuous function $f : I \to \mathbb{R}$ on a compact interval $I \subset \mathbb{R}$ is the uniform limit of a sequence of polynomials. The direct analogue of this to the complex setting (obtained by replacing $\mathbb{R}$ with $\mathbb{C}$, I with a compact set $K \subset \mathbb{C}$ and real polynomials with complex polynomials) is false, even if we make a suitable holomorphicity assumption on f. Construct, for any given compact set $K \subset \mathbb{C}$ with $\mathbb{C} \setminus K$ not connected, a function f that is holomorphic on an open set containing K such that f is not the uniform limit on K of a sequence of complex polynomials. [Hint: you may wish to generalise the idea of Q 13(ii) in sheet 1 for the construction, and use the global maximum principle to prove it works.] Look up, on the other hand, Runge's theorem and Mergelyan's theorem!

2. (a) Draw a (convincing!) picture of a domain Ω and a closed curve γ in Ω such that γ is homologous to zero in Ω but is not null-homotopic in Ω. (The reverse direction, as proved in lecture, is always true, i.e. if γ is null-homotopic in Ω, then it is homologous to zero in Ω).
(b) Let U be a domain with the property that *every* closed piecewise C^1 curve in U is homologous to zero in U.
(i) Use Cauchy's theorem to show that if f is a nowhere vanishing holomorphic function on U, then f admits a holomorphic square-root (i.e. there is a holomorphic function h such that $h^2(z) = f(z)$ for every $z \in U$.)
(ii) The key ingredient of a standard proof of the Riemann mapping theorem is to show that whenever a domain has the property that every nowhere zero holomorphic function on it admits a holomorphic square-root, then it is homeomorphic to the open unit disk (the non-trivial case of this being when the domain is not equal to $\mathbb{C}$, in which case the homeomorphism is in fact shown to be a conformal map). Assuming this, deduce that U is simply connected, i.e. has the property that every closed curve in U is null-homotopic in U. Thus a domain U is simply connected if and only if every closed curve in U is homologous to zero in U.

3. (a) Let $0 \le r < R \le \infty$, $A = \{r < |z| < R\}$ and let f be holomorphic on A. Show that there is a *unique* decomposition $f = f_1 + f_2$ such that f_1 is holomorphic on $\{|z| < R\}$, f_2 is holomorphic on $\{|z| > r\}$ and $f_2(z) \to 0$ as $z \to \infty$.
(b) How does this extend to the case when A is a (bounded) domain between two non-concentric disjoint circles? What about a domain bounded by three disjoint circles?

4. Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a,R)$, and $|w-a| < r < R$, then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D(a,r)} \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

5. Let U be a domain in $\mathbb{C} \approx \mathbb{R}^2$ and let $u : U \to \mathbb{R}$ be a C^2 harmonic function.
(a) If U is simply connected, show that there is a holomorphic function $f : U \to \mathbb{C}$ such that $u = \operatorname{Re} f$. (Hint: consider $g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$).
(b) If $U = D(a,r)$, show that

$$\sup_{z \in D(a,r/2)} |Du(z)| \leq \frac{C}{r} \sup_{z \in D(a,r)} |u|$$

where C is a fixed constant independent of u, r and a. Here $Du = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$ and $|Du| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$. (Hint: we can assume without loss of generality that $a = 0$, $r = 1$ and $\sup_{D(0,1)} |u| = 1$; why? Now apply the Cauchy derivative estimate to a suitable holomorphic function).
(c) Now suppose $U = D(0,1) \setminus \{0\}$. If u has a continuous extension to $D(0,1)$, show that the point $z = 0$ is a removable singularity of u, i.e. show that the extended function is C^2 and harmonic in $D(0,1)$. (Hint: start by using (b) to analyse the behaviour of g on approach to 0).
(d) In (c), it suffices to assume that u is bounded on U. How can you generalise your proof to this case?
(e) In (c), what if we only assume $\lim_{z\to 0} |z||u(z)| = 0$? Compare with the case of holomorphic $f : D(0,1) \setminus \{0\} \to \mathbb{C}$ satisfying $\lim_{z\to 0} |z||f(z)| = 0$.

6. Evaluate the following integrals:

$$(a) \quad \int_0^{\pi} \frac{d\theta}{4 + \sin^2\theta}; \qquad (b) \quad \int_0^{\infty} \sin x^2 \, dx;$$

$$(c) \quad \int_0^{\infty} \frac{x^2}{(x^2+4)^2(x^2+9)} \, dx; \qquad (d) \quad \int_0^{\infty} \frac{\log(x^2+1)}{x^2+1} \, dx.$$

7. For $\alpha \in (-1,1)$ with $\alpha \neq 0$, compute $\int_0^{\infty} \frac{x^\alpha}{x^2+x+1} \, dx$.
8. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.
(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2+1}$ around γ_N, show that $\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1+\pi \coth\pi}{2}$.
(iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n/(n^2+1)$.
9. Let f be holomorphic in an open set U except at a point $a \in U$ and at a sequence of points $a_n \in U$ with $a_n \neq a$ and $a_n \to a$. Suppose that each a_n is a pole of f. Note that a is then a non-isolated singularity. (i) Give an explicit example of such a function f, points a_n and a. (ii) What can you say (in general) about the image $f(U \setminus \{a, a_1, a_2, \ldots\})$?
10. Let f_n be a sequence of holomorphic functions on a domain U converging locally uniformly to a function $f : U \to \mathbb{C}$. If $f_n(z) \neq 0$ for each n and each $z \in U$, show that either $f(z) = 0$ for all $z \in U$ or $f(z) \neq 0$ for all $z \in U$. What if we allow each f_n to have at most k zeros in U for some fixed positive integer k independent of n?
11. Establish the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and let $A = \max\{|a_i| : 0 \leq i \leq n-1\}$. Then $p(z)$ has n roots (counted with multiplicity) in the disk $|z| < A + 1$.
12. If $f : U \to \mathbb{C}$ is holomorphic and one-to-one, show that $f'(z) \neq 0$ for all $z \in U$.

13. (i) Show that $z^4 + 12z + 1 = 0$ has exactly three zeros with $1 < |z| < 4$.
(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \bigm| \mathrm{Re}(z) < 0 \,\}$.
(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{\, z : \ |z| = 3/2 \,\}$.

14. Let f be a function which is analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \le |z|^k$ for all z with $|z|$ sufficiently large, then f is a rational function (i.e. a quotient of two polynomials).

15. Show that the equation $z \sin z = 1$ has only real solutions. [Hint: Find the number of real roots in the interval $[-(n+1/2)\pi, (n+1/2)\pi]$ and compare with the number of zeros of $z \sin z - 1$ is a square box $\{|\mathrm{Re}\, z|, |\mathrm{Im}\, z| < (n+1/2)\pi\}$.]

16. Let U be a domain, let $f : U \to \mathbb{C}$ be holomorphic and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) = \frac{1}{2\pi i} \int_{\partial\, D(a,r)} \frac{z f'(z)}{f(z) - w}\, dz$$

defines a holomorphic function g in a neighbourhood of $f(a)$ which is inverse to f.

The following integrals are *not* part of the question sheet, but are provided as a starting point for revision, or for the enthusiast.

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(a^2+x^2)}\, dx$ where $a,\ m \in \mathbb{R}^+$;

(2) $\displaystyle\int_0^{2\pi} \frac{\cos^3 3t}{1 - 2a\cos t + a^2}\, dt$ where $a \in (0,1)$;

(3) $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx$ ("dog-bone" contour);

(4) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx}\, dx$ where $t \in \mathbb{R}$.

(5) By integrating $z/(a - e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^{\pi} \frac{x \sin x}{1 - 2a\cos x + a^2}\, dx = \frac{\pi}{a} \log(1+a)$$

for every $a \in (0,1)$.

(6) Assuming $\alpha \ge 0$ and $\beta \ge 0$ prove that

$$\int_0^{\infty} \frac{\cos \alpha x - \cos \beta x}{x^2}\, dx = \frac{\pi}{2}(\beta - \alpha),$$

and deduce the value of

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$