COMPLEX ANALYSIS EXAMPLES 2, LENT 2022

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1. Use the Cauchy integral formula to compute $\int_{\partial D(0,2)} \frac{dz}{z^2+1}$ and $\int_{\partial D(0,2)} \frac{dz}{z^2-1}$. Are the answers an accident? Formulate and prove a result for a polynomial with n distinct roots.

2. (i) Use the Cauchy integral formula to compute

$$\int_{\partial D(0,1)} \frac{e^{\alpha z}}{2z^2 - 5z + 2}\, dz$$

where $\alpha \in \mathbb{C}$.
(ii) By considering suitable complex integrals, show that if $r \in (0,1)$,

$$\int_0^\pi \frac{\cos n\theta}{1 - 2r\cos\theta + r^2}\, d\theta = \frac{\pi r^n}{1 - r^2} \qquad \text{and} \qquad \int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\, d\theta = 2\pi.$$

3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Prove that if any one of the following conditions hold, then f is constant:
(i) $f(z)/z \to 0$ as $|z| \to \infty$.
(ii) There exists $b \in \mathbb{C}$ and $\varepsilon > 0$ such that for every $z \in \mathbb{C}$, $|f(z) - b| > \varepsilon$.
(iii) $f = u + iv$ and $|u(z)| > |v(z)|$ for all $z \in \mathbb{C}$.

4. Let $f\colon D(a,r) \to \mathbb{C}$ be holomorphic, and suppose that $z = a$ is a local maximum for $\mathrm{Re}(f)$. Show that f is constant. What if $z = a$ is a local minimum for $\mathrm{Re}(f)$? What if $z = a$ is a local minimum for $|f|$? Do your answers change if additionally $f(a) \neq 0$?

5. (i) Let f be an entire function. Show that f is a polynomial, of degree $\leq k$, if and only if there is a constant M for which $|f(z)| < M(1 + |z|)^k$ for all z.
(ii) Show that an entire function f is a polynomial of positive degree if and only if $|f(z)| \to \infty$ as $|z| \to \infty$.

6. Recall that the (global) maximum principle says that if U is a bounded open subset of $\mathbb{C}$ and if f is continuous on $\overline{U}$ (the closure of U in $\mathbb{C}$) and holomorphic in U, then $\sup_{\overline{U}} |f| = \sup_{\partial U} |f|$. Does this hold if U is not assumed to be bounded?

7. (i) (Schwarz's Lemma) Let f be holomorphic on $D(0,1)$, satisfying $|f(z)| \leq 1$ and $f(0) = 0$. By applying the maximum principle to $f(z)/z$, show that $|f(z)| \leq |z|$. Show also that if $|f(w)| = |w|$ for some $w \neq 0$ then $f(z) = cz$ for some constant c.
(ii) Use Schwarz's Lemma to prove that any conformal equivalence from $D(0,1)$ to itself is given by a Möbius transformation.

8. Let $f : \partial D(0,1) \to \mathbb{C}$ be continuous. Show that the function $g(w) = \frac{1}{2\pi i} \int_{\partial D(0,1)} \frac{f(z)}{z-w} dz$ is holomorphic in $\mathbb{C} \setminus \partial D(0,1)$. Must there be a point $a \in \partial D(0,1)$ such that $g(w) \to f(a)$ as $w \to a, w \in D(0,1)$? If there is a sequence of polynomials (p_n) converging uniformly to f on $\partial D(0,1)$, show that there is a continuous function h on $\overline{D(0,1)}$ such that h is holomorphic on $D(0,1)$ and $h(w) = f(w)$ for every $w \in \partial D(0,1)$. Must it be true that $p_n(w) \to h(w)$ for every $w \in \overline{D(0,1)}$? How does h relate to g?

9. (i) Let f be an entire function such that for every positive integer n, $f(1/n) = 1/n$. Show that $f(z) = z$.
(ii) Let f be an entire function with $f(n) = n^2$ for every $n \in \mathbb{Z}$. Must $f(z) = z^2$?
(iii) Let f be holomorphic on $D(0,2)$. Show that for some integer $n > 0$, $f(1/n) \neq 1/(n+1)$.

10. (i) Show that the power series $\sum_{n=1}^{\infty} z^{n!}$ defines an analytic function f on $D(0,1)$. Show that f cannot be analytically continued to any domain which properly contains $D(0,1)$. [Hint: consider $z = \exp(2\pi i p/q)$ with p/q rational.]
(ii) It is true, but it takes some effort to show, that *if $U \subset \mathbb{C}$ is a domain and $(a_n)_{n=1}^{\infty}$ is a sequence of distinct points in U such that the set $\{a_n : n = 1, 2, 3, \ldots\}$ has no limit point in U, then there is a holomorphic function on U whose only zeros are the points a_n, $n = 1, 2, 3, \ldots$.* Use this fact to show that if $U \subset \mathbb{C}$ is any bounded domain, then there is a holomorphic function $f : U \to \mathbb{C}$ such that f has no analytic continuation to any domain which properly contains U.

11. (i) Let $w \in \mathbb{C}$, and let γ, δ: $[0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve

$$\sigma(t) = \frac{\delta(t) - w}{\gamma(t) - w}$$

about the origin, show that $I(\gamma; w) = I(\delta; w)$.
(ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w,r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w,r)$.
(iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

12. Find the Laurent expansion (in powers of z) of $1/(z^2 - 3z + 2)$ in each of the regions:

$$\{z : |z| < 1\}; \quad \{z : 1 < |z| < 2\}; \quad \{z : |z| > 2\}.$$

13. Classify the singularities of each of the following functions:

$$\frac{z}{\sin z}, \qquad \sin\frac{\pi}{z^2}, \qquad \frac{1}{z^2} + \frac{1}{z^2+1}, \qquad \frac{1}{z^2}\cos\left(\frac{\pi z}{z+1}\right).$$

14. (Casorati-Weierstrass theorem) Let f be holomorphic on $D(a,R) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a,R)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes *every* complex value except possibly one.]

15. Let f be a holomorphic function on $D(a,R) \setminus \{a\}$. Show that if f has a non-removable singularity at $z = a$, then the function $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\operatorname{Re} f(z) < M$ for $z \in D(a,R) \setminus \{a\}$, then f has a removable singularity at $z = a$.