COMPLEX ANALYSIS EXAMPLES 1, LENT 2022

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1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a real linear map. Regarding T as a map from $\mathbb{C}$ into $\mathbb{C}$ by identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, show that there exist unique complex numbers A, B such that for every $z \in \mathbb{C}$, $T(z) = Az + B\bar{z}$. Show that T is complex differentiable if and only if $B = 0$.

2. (i) Let $f : D \to \mathbb{C}$ be a holomorphic function defined on a domain D. Show that f is constant if any one of its real part, imaginary part, modulus or argument is constant.

(ii) Find all holomorphic functions on $\mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$ where u and v are both real valued.

(iii) Find all holomorphic functions on $\mathbb{C}$ with real part $x^3 - 3xy^2$.

3. (i) Define $f : \mathbb{C} \to \mathbb{C}$ by $f(0) = 0$, and

$$f(z) = \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2} \quad \text{for } z = x + iy \neq 0.$$

Show that f satisfies the Cauchy-Riemann equations at 0. Show further that f is continuous everywhere but is not differentiable at 0.

(ii) Define $g : \mathbb{C} \to \mathbb{C}$ by $g(0) = 0$ and $g(z) = e^{-\frac{1}{z^4}}$ for $z \neq 0$. Show that g satisfies the Cauchy–Riemann equations everywhere, but is not continuous (hence also not differentiable) at 0.

4. (i) Define the differential operators $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$. Let $U \subset \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be a C^1 function in the sense that $\mathrm{Re}\,(f)$ and $\mathrm{Im}\,(f)$ are each C^1 on U (with U taken as an open subset of $\mathbb{R}^2$). Prove that f is holomorphic iff $\partial f/\partial \bar{z} = 0$. Show that

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the usual Laplacian in $\mathbb{R}^2$.

(ii) Let $f : U \to V$ be holomorphic and let $g : V \to \mathbb{C}$ be harmonic. Show that the composition $g \circ f$ is harmonic.

5. (i) Denote by Log the principal branch of the logarithm. If $z \in \mathbb{C}$, show that $n\,\mathrm{Log}(1 + z/n)$ is defined if n is sufficiently large, and that it tends to z as n tends to ∞. Deduce that for any $z \in \mathbb{C}$,

$$\lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n = e^z.$$

(ii) Defining $z^\alpha = \exp(\alpha\,\mathrm{Log}\,z)$, where Log is the principal branch of the logarithm and $z \notin \mathbb{R}_{\leq 0}$, show that $\frac{d}{dz}(z^\alpha) = \alpha z^{\alpha-1}$. Does $(zw)^\alpha = z^\alpha w^\alpha$ always hold?

6. Prove that each of the following series converges uniformly on compact (i.e. closed and bounded) subsets of the given domains in $\mathbb{C}$:

$$(a)\ \sum_{n=1}^{\infty}\sqrt{n}e^{-nz} \quad \text{on } \{\, z :\ 0 < \mathrm{Re}(z)\,\}; \qquad (b)\ \sum_{n=1}^{\infty}\frac{2^n}{z^n + z^{-n}} \quad \text{on } \left\{ z :\ 0 < |z| < \frac{1}{2} \right\}.$$

7. Find conformal equivalences between the following pairs of domains:

(i) the sector $\{z \in \mathbb{C} : -\pi/4 < \arg(z) < \pi/4\}$ and the open unit disc $D(0,1)$;

(ii) the lune $\{z \in \mathbb{C} : |z-1| < \sqrt{2} \text{ and } |z+1| < \sqrt{2}\}$ and $D(0,1)$;

(iii) the strip $S = \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1\}$ and the quadrant $Q = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$.

By considering a suitable solution of Laplace's equation $u_{xx} + u_{yy} = 0$ on S, find a non-constant harmonic function φ on Q which extends continuously to $\overline{Q} \setminus \{0\}$ with constant values on each of the two components of $\partial Q \setminus \{0\}$. ($\varphi$ need not be continuous at the origin. Here $\overline{Q}$ denotes the closure of Q in $\mathbb{R}^2$ and $\partial Q = \overline{Q} \setminus Q$.)

8. (i) Show that the most general Möbius transformation which maps the unit disk onto itself has the form $z \mapsto \lambda \dfrac{z-a}{\bar{a}z-1}$, with $|a| < 1$ and $|\lambda| = 1$. [*Hint: first show that these maps form a group.*]

(ii) Find a Möbius transformation taking the region between the circles $\{|z| = 1\}$ and $\{|z-1| = 5/2\}$ to an annulus $\{1 < |z| < R\}$. [*Hint: a circle can be described by an equation of the shape* $|z-a|/|z-b| = \ell$.]

(iii) Find a conformal map from an infinite strip onto an annulus. Can such a map ever be a Möbius transformation?

9. Let $U \subset \mathbb{C}$ be open and let $f = u + iv : U \to \mathbb{C}$. Suppose that u and v are C^1 on U as real functions of the real variables x, y where $x + iy \in U$. Let $w \in U$ and suppose that the map f is angle-preserving at w in the following sense: for any two C^1 curves $\gamma_1, \gamma_2 : (-1,1) \to U$ with $\gamma_j(0) = w$ and $\gamma_j'(0) \neq 0$ for $j = 1, 2$, the curves $\alpha_j = f \circ \gamma_j = u \circ \gamma_j + iv \circ \gamma_j$ satisfy $\alpha_j'(0) \neq 0$ and $\arg \frac{\alpha_1'(0)}{\gamma_1'(0)} = \arg \frac{\alpha_2'(0)}{\gamma_2'(0)}$. Show that f is complex differentiable at w with $f'(w) \neq 0$. [You may find it useful to employ the operator $\frac{\partial}{\partial \bar{z}}$ in Q4].

10. Use the (real) inverse function theorem (from the Analysis & Topology course) to prove the following holomorphic inverse function theorem: if $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic and $f'(z_0) \neq 0$ for some $z_0 \in U$, then there is an open neighborhood V of z_0 and an open neighborhood W of $f(z_0)$ such that $f|_V : V \to W$ is a bijection with holomorphic inverse. [Use the fact that holomorphic functions are C^1, i.e. have C^1 real and imaginary parts; we will prove this—in fact that holomorphic functions are infinitely differentiable—later in the course.]

11. Calculate $\int_\gamma z \sin z \, dz$ when γ is the straight line joining 0 to i.

12. Show that the following functions do not have antiderivatives (i.e. functions of which they are the derivatives) on the domains indicated:

$$(a) \quad \frac{1}{z} - \frac{1}{z-1} \quad (0 < |z| < 1); \qquad (b) \quad \frac{z}{1+z^2} \quad (1 < |z| < \infty).$$

13. Does there exist a sequence of polynomials $p_n(z)$ converging uniformly to $1/z$ on: (i) the disk $\{z \in \mathbb{C} : |z-1| < 1/2\}$? (ii) the annulus $\{z \in \mathbb{C} : 1/2 < |z| < 1\}$?

14. Let $U \subset \mathbb{C}$ be a domain, and let $u : U \to \mathbb{R}$ be a C^2 harmonic function. Show that if $z_0 \in U$ then for any disk $D = D(z_0, \rho) \subset U$, there is a holomorphic function $f : D \to \mathbb{C}$ such that $u = \mathrm{Re}\,(f)$ on D. Show by an example that this need not hold globally, i.e. that there need not exist holomorphic $f : U \to \mathbb{C}$ such that $u = \mathrm{Re}\,(f)$ on all of U.