COMPLEX ANALYSIS EXAMPLES 3

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Use the residue theorem to give a proof of Cauchy's derivative formula: if f is holomorphic on $D(a, R)$, and $|w - a| < r < R$, then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

2. Let $g(z) = p(z)/q(z)$ be a rational function with $\deg(q) \geq \deg(p) + 2$. Show that the sum of the residues of g at all its poles equals zero.

3. Evaluate the following integrals:

$(a)\quad \displaystyle\int_0^\pi \frac{d\theta}{4+\sin^2\theta};$ $\qquad (b)\quad \displaystyle\int_0^\infty \sin x^2\, dx;$

$(c)\quad \displaystyle\int_0^\infty \frac{x^2}{(x^2+4)^2(x^2+9)}\, dx;$ $\qquad (d)\quad \displaystyle\int_0^\infty \frac{\log(x^2+1)}{x^2+1}\, dx.$

4. For $\alpha \in (-1, 1)$ with $\alpha \neq 0$, compute

$$\int_0^\infty \frac{x^\alpha}{x^2+x+1}\, dx.$$

5. Establish the following refinement of the Fundamental Theorem of Algebra. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and let $A = \max\{|a_i| :\ 0 \leq i \leq n-1\}$. Then $p(z)$ has n roots (counted with multiplicity) in the disk $|z| < A + 1$.

6. Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for such z. Hence sketch the curve $p \circ \gamma$, where $\gamma(t) = e^{2\pi i t}$ and use your sketch to determine the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = x$ for each real number x.

7. (i) For a positive integer N, let γ_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. Show that there exists $C > 0$ such that for every N, $|\cot \pi z| < C$ on γ_N.

(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2+1}$ around γ_N, show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1 + \pi \coth\pi}{2}.$$

(iii) Evaluate $\sum_{n=0}^{\infty} (-1)^n/(n^2+1)$.

8. (i) Show that $z^4 + 12z + 1 = 0$ has exactly three zeros with $1 < |z| < 4$.

(ii) Prove that $z^5 + 2 + e^z$ has exactly three zeros in the half-plane $\{\, z \mid \operatorname{Re}(z) < 0 \,\}$.

(iii) Show that the equation $z^4 + z + 1 = 0$ has one solution in each quadrant. Prove that all solutions lie inside the circle $\{ z : |z| = 3/2 \}$.

9. (i) Let $w \in \mathbb{C}$, and let γ, $\delta : [0,1] \to \mathbb{C}$ be closed curves such that for all $t \in [0,1]$, $|\gamma(t) - \delta(t)| < |\gamma(t) - w|$. By computing the winding number of the closed curve

$$\sigma(t) = \frac{\delta(t) - w}{\gamma(t) - w}$$

about the origin, show that $I(\gamma; w) = I(\delta; w)$.
(ii) If $w \in \mathbb{C}$, $r > 0$, and γ is a closed curve which does not meet $D(w, r)$, show that $I(\gamma; w) = I(\gamma; z)$ for every $z \in D(w, r)$.
(iii) Deduce that if γ is a closed curve in $\mathbb{C}$ and U is the complement of (the image of) γ, then the function $w \mapsto I(\gamma; w)$ is a locally constant function on U.

10. Show that the equation $z \sin z = 1$ has only real solutions. [Hint: Find the number of real roots in the interval $[-(n+1/2)\pi, (n+1/2)\pi]$ and compare with the number of zeros of $z \sin z - 1$ is a square box $\{|\mathrm{Re}\, z|, |\mathrm{Im}\, z| < (n+1/2)\pi\}$.]

11. Let U be a domain, let $f : U \to \mathbb{C}$ be holomorphic and suppose $a \in U$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small,

$$g(w) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{z f'(z)}{f(z) - w}\, dz$$

defines a holomorphic function g in a neighbourhood of $f(a)$ which is inverse to f.

The following integrals are *not* part of the question sheet, but are provided as a starting point for revision, or for the enthusiast.

(1) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx}{x(a^2 + x^2)}\, dx$ where $a,\ m \in \mathbb{R}^+$;

(2) $\displaystyle\int_{0}^{2\pi} \frac{\cos^3 3t}{1 - 2a\cos t + a^2}\, dt$ where $a \in (0,1)$;

(3) $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx$ ("dog-bone" contour);

(4) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx}\, dx$ where $t \in \mathbb{R}$.

(5) By integrating $z/(a - e^{-iz})$ round the rectangle with vertices $\pm\pi$, $\pm\pi + iR$, prove that

$$\int_0^{\pi} \frac{x \sin x}{1 - 2a\cos x + a^2}\, dx = \frac{\pi}{a} \log(1 + a)$$

for every $a \in (0,1)$.

(6) Assuming $\alpha \geq 0$ and $\beta \geq 0$ prove that

$$\int_0^{\infty} \frac{\cos \alpha x - \cos \beta x}{x^2}\, dx = \frac{\pi}{2}(\beta - \alpha),$$

and deduce the value of

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$