COMPLEX ANALYSIS EXAMPLES 2

G.P. Paternain Lent 2017

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at g.p.paternain@dpmms.cam.ac.uk.

1. Use the Cauchy integral formula to compute \( \int_{|z|=2} \frac{dz}{z^2+1} \) and \( \int_{|z|=2} \frac{dz}{z^2-1} \). Are the answers an accident? Formulate and prove a result for a polynomial with \( n \) distinct roots.

2. (i) Use the Cauchy integral formula to compute

\[
\int_{|z|=1} \frac{e^{\alpha z}}{2z^2 - 5z + 2} dz
\]

where \( \alpha \in \mathbb{C} \).

(ii) By considering suitable complex integrals, show that if \( r \in (0, 1) \),

\[
\int_0^\pi \frac{\cos n\theta}{1 - 2r \cos \theta + r^2} d\theta = \frac{\pi r^n}{1 - r^2} \quad \text{and} \quad \int_0^{2\pi} \cos(\cos \theta) \cosh(\sin \theta) \, d\theta = 2\pi.
\]

3. Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. Prove that if any one of the following conditions hold, then \( f \) is constant:

(i) \( f(z)/z \to 0 \) as \( |z| \to \infty \).

(ii) There exists \( b \in \mathbb{C} \) and \( \varepsilon > 0 \) such that for every \( z \in \mathbb{C} \), \( |f(z) - b| > \varepsilon \).

(iii) \( f = u + iv \) and \( |u(z)| > |v(z)| \) for all \( z \in \mathbb{C} \).

4. Let \( f : D(a, r) \to \mathbb{C} \) be holomorphic, and suppose that \( z = a \) is a local maximum for \( \text{Re}(f) \). Show that \( f \) is constant.

5. (i) Let \( f \) be an entire function. Show that \( f \) is a polynomial, of degree \( \leq k \), if and only if there is a constant \( M \) for which \( |f(z)| < M(1 + |z|)^k \) for all \( z \).

(ii) Show that an entire function \( f \) is a polynomial of positive degree if and only if \( |f(z)| \to \infty \) as \( |z| \to \infty \).

(iii) Let \( f \) be a function which is analytic on \( \mathbb{C} \) apart from a finite number of poles. Show that if there exists \( k \) such that \( |f(z)| \leq |z|^k \) for all \( z \) with \( |z| \) sufficiently large, then \( f \) is a rational function (i.e. a quotient of two polynomials).

6. (i) (Schwarz’s Lemma) Let \( f \) be analytic on \( D(0, 1) \), satisfying \( |f(z)| \leq 1 \) and \( f(0) = 0 \). By applying the maximum principle to \( f(z)/z \), show that \( |f(z)| \leq |z| \). Show also that if \( |f(w)| = |w| \) for some \( w \neq 0 \) then \( f(z) = cz \) for some constant \( c \).

(ii) Use Schwarz’s Lemma to prove that any conformal equivalence from \( D(0, 1) \) to itself is given by a M"obius transformation.

7. (i) Let \( f \) be an entire function such that for every positive integer \( n \), \( f(1/n) = 1/n \). Show that \( f(z) = z \).

(ii) Let \( f \) be an entire function with \( f(n) = n^2 \) for every \( n \in \mathbb{Z} \). Must \( f(z) = z^2 \)?

(iii) Let \( f \) be holomorphic on \( D(0, 2) \). Show that for some integer \( n > 0 \), \( f(1/n) \neq 1/(n+1) \).
8. (i) Give an example of an infinitely differentiable function $f : (-1, 1) \to \mathbb{R}$ which can be extended to a holomorphic function on a domain $U \subset \mathbb{C}$ containing $(-1, 1)$, but for which one cannot take $U$ to be the open unit disc $D(0, 1)$.

(ii) Give an example of an infinitely differentiable function $f : (-1, 1) \to \mathbb{R}$ which is not the restriction of any holomorphic function defined on a domain $U \subset \mathbb{C}$ containing $(-1, 1)$.

(iii) Prove that the integral $\int_0^\infty e^{-zt} \sin(t) \, dt$ converges for $\text{Re}(z) > 0$ and defines a holomorphic function in that half-plane. Prove furthermore that the resulting holomorphic function admits an analytic continuation to $\mathbb{C} \setminus \{\pm i\}$.

(iv) Show that the power series $\sum_{n=1}^\infty z^n$ defines an analytic function $f$ on $D(0, 1)$. Show that $f$ cannot be analytically continued to any domain which properly contains $D(0, 1)$.

[Hint: consider $z = \exp(2\pi i p/q)$ with $p/q$ rational.]

9. Find the Laurent expansion (in powers of $z$) of $1/(z^2 - 3z + 2)$ in each of the regions:

- $\{z \mid |z| < 1\}$;
- $\{z \mid 1 < |z| < 2\}$;
- $\{z \mid |z| > 2\}$.

10. Classify the singularities of each of the following functions:

\[
\frac{z}{\sin z}, \quad \sin \frac{\pi}{z^2}, \quad \frac{1}{z^2 + \frac{1}{z^2 + 1}}, \quad \frac{1}{z^2} \cos \left( \frac{\pi z}{z + 1} \right).
\]

11. (Casorati-Weierstrass theorem) Let $f$ be holomorphic on $D(a, R) \setminus \{a\}$ with an essential singularity at $z = a$. Show that for any $b \in \mathbb{C}$, there exists a sequence of points $z_n \in D(a, R)$ with $z_n \neq a$ such that $z_n \to a$ and $f(z_n) \to b$ as $n \to \infty$.

Find such a sequence when $f(z) = e^{1/z}$, $a = 0$ and $b = 2$.

[A much harder theorem of Picard says that in any neighbourhood of an essential singularity, an analytic function takes every complex value except possibly one.]

12. Let $f$ be a holomorphic function on $D(a, R) \setminus \{a\}$. Show that if $f$ has a non-removable singularity at $z = a$, then the function $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists $M$ such that $\text{Re} f(z) < M$ for $z \in D(a, R) \setminus \{a\}$, then $f$ has a removable singularity at $z = a$. 