ANALYSIS II EXAMPLES 1

G.P. Paternain Mich. 2003

The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. The questions are based on the example sheets I gave last year, but I have made a few changes.

1. Define $f_n : [0,2] \to \mathbb{R}$ by

$$\begin{aligned} f_n(x) &= 1 - n|x - n^{-1}| \qquad && \text{for } |x - n^{-1}| \leq n^{-1}, \\ f_n(x) &= 0 && \text{otherwise.} \end{aligned}$$

Show that the f_n are continuous and sketch their graphs. Show that f_n converges pointwise on $[0,2]$ to the zero function but not uniformly.

2. Let f and g be uniformly continuous real-valued functions on a set E.
(i) Show that the (pointwise) sum $f + g$ is uniformly continuous on E, as also is λf for any real constant λ.
(ii) Is the product fg necessarily uniformly continuous on E? Give a proof or counter-example as appropriate.

3. Consider the functions $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = n^p x \exp(-n^q x)$ where p, q are positive constants.
(i) Show that f_n converges pointwise on $[0,1]$, for any p and q.
(ii) Show that if $p < q$ then f_n converges uniformly on $[0,1]$.
(iii) Show that if $p \geq q$ then f_n does not converge uniformly on $[0,1]$. Does f_n converge uniformly on $[0, 1-\epsilon]$? Does f_n converge uniformly on $[\epsilon, 1]$? [Here $0 < \epsilon < 1$; you should justify your answers.]

4. Let $f_n(x) = n^\alpha x^n (1-x)$, where α is a real constant.
(i) For which values of α does $f_n(x) \to 0$ pointwise on $[0,1]$?
(ii) For which values of α does $f_n(x) \to 0$ uniformly on $[0,1]$?
(iii) For which values of α does $\int_0^1 f_n(x)\,dx \to 0$?
(iv) For which values of α does $f_n'(x) \to 0$ pointwise on $[0,1]$?
(v) For which values of α does $f_n'(x) \to 0$ uniformly on $[0,1]$?

5. Consider the sequence of functions $f_n\colon (\mathbb{R} \setminus \mathbb{Z}) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{m=0}^{n} (x-m)^{-2} \ .$$

(i) Show that f_n converges pointwise on $\mathbb{R} \setminus \mathbb{Z}$ to a function f.
(ii) Show that f_n does not converge uniformly on $\mathbb{R} \setminus \mathbb{Z}$.
(iii) Why can we nevertheless conclude that the limit function f is continuous, and indeed differentiable, on $\mathbb{R} \setminus \mathbb{Z}$?

6. Suppose f_n is a sequence of continuous functions from a bounded closed interval $[a,b]$ to $\mathbb{R}$, and that f_n converges pointwise to a continuous function f.
(i) If f_n converges uniformly to f, and (x_m) is a sequence of points of $[a,b]$ converging to a limit x, show that $f_n(x_n) \to f(x)$. [Careful — this is not quite as easy as it looks!]

(ii) If f_n does **not** converge uniformly, show that we can find a convergent sequence $x_n \to x$ in $[a, b]$ such that $f_n(x_n)$ does not converge to $f(x)$. [Hint: Bolzano–Weierstrass.]

7. (i) Suppose f is defined and differentiable on a (bounded or unbounded) interval $E \subseteq \mathbb{R}$, and that its derivative f' is bounded on E. Use the Mean Value Theorem to show that f is uniformly continuous on E.

(ii) Give an example of a function f which is (uniformly) continuous on $[0, 1]$, and differentiable at every point of $[0, 1]$ (here we interpret $f'(0)$ as the 'one-sided derivative' $\lim_{h\to 0^+}((f(h) - f(0))/h)$, and similarly for $f'(1)$), but such that f' is unbounded on $[0, 1]$. [Hint: last year you probably saw an example of an everywhere differentiable function whose derivative is discontinuous; you will need to 'tweak' it slightly.]

8. Suppose that f is continuous on $[0, \infty)$ and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Is f necessarily uniformly continuous on $[0, \infty)$? Give a proof or a counterexample as appropriate.

9. Which of the following functions f are (a) uniformly continuous, (b) bounded on $[0, \infty)$?
(i) $f(x) = \sin x^2$.
(ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$.
(iii) $f(x) = (\sin x^3)/(x + 1)$.

10. Let f be a bounded function defined on a set $E \subseteq \mathbb{R}$, and for each positive integer n let g_n be the function defined on E by

$$g_n(x) = \sup\{|f(y) - f(x)| : y \in E, |y - x| < 1/n\} .$$

Show that f is uniformly continuous on E if and only if $g_n \to 0$ uniformly on E as $n \to \infty$.

11. (i) Show that if (f_n) is a sequence of uniformly continuous functions on $\mathbb{R}$, and $f_n \to f$ uniformly on $\mathbb{R}$, then f is uniformly continuous.

(ii) Give an example of a sequence of uniformly continuous functions f_n on $\mathbb{R}$, such that f_n converges pointwise to a continuous function f, but f is not uniformly continuous. [Hint: choose the limit function f first, and then take the f_n to be a sequence of 'approximations' to it.]

***12**. Define $\varphi(x) = |x|$ for $x \in [-1, 1]$ and extend the definition of $\varphi(x)$ to all real x by requiring that

$$\varphi(x + 2) = \varphi(x).$$

(i) Show that $|\varphi(s) - \varphi(t)| \leq |s - t|$ for all s and t.
(ii) Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Prove that f is well defined and continuous.
(iii) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2}\, 4^{-m}$ where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. Prove that

$$\left|\frac{f(x + \delta_m) - f(x)}{\delta_m}\right| \geq \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x. Hence there exists a real continuous function on the real line which is nowhere differentiable.

***13**. *A space-filling curve (Exercise 14, Chapter 7 of Rudin's book).* Let f be a continuous real function on $\mathbb{R}$ with the following properties: $0 \leq f(t) \leq 1$, $f(t + 2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & \text{for } t \in [0, 1/3]; \\ 1 & \text{for } t \in [2/3, 1]. \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1} t), \qquad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n} t).$$

Prove that Φ is continuous and that Φ maps $I = [0,1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. In fact, show that Φ maps the Cantor set onto I^2.

Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.