ANALYSIS II—EXAMPLES 4 Michaelmas 2025

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1. Quickies: (a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}^n$. If the directional derivatives $D_u f(a)$ exist for all directions $u \in \mathbb{R}^2$ and if $D_u f(a)$ depends linearly on u, does it follow that f is differentiable at a?

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. If f is differentiable at $0 \in \mathbb{R}^2$, and if the partial derivatives of f exist in a neighbourhood of 0, does it follow that one partial derivative is continuous at 0?

(c) Let $f\colon [a,b] \to \mathbb{R}^2$ be continuous, and differentiable on (a,b). Does it follow that there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$?

(d) Let $F\colon [0,1] \times \mathbb{R}^m \to \mathbb{R}$ be continuous and $a = (a_0, \ldots, a_{m-1}) \in \mathbb{R}^m$. Suppose that F is uniformly Lipschitz in the $\mathbb{R}^m$ variables near a, i.e. for some constant K and an open subset U of $\mathbb{R}^m$ containing a, $|F(t,x) - F(t,y)| \leq K\|x-y\|$ for all $t \in [0,1]$, $x, y \in U$. Use the Picard–Lindelöf existence theorem for first order ODE systems to show that there is an $\epsilon > 0$ such that, writing $f^{(j)}$ for the jth derivative of f, the mth order initial value problem

$$f^{(m)}(t) = F(t, f(t), f^{(1)}(t), \ldots, f^{(m-1)}(t)) \quad \text{for} \quad t \in [0,\epsilon);$$

$$f^{(j)}(0) = a_j \quad \text{for} \quad 0 \leq j \leq m-1$$

has a unique C^m solution $f : [0,\epsilon) \to \mathbb{R}$.

2. (*The operator norm*). Show that any linear map $A\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Let $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ be the vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Recall that the operator norm on $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ is defined by $\|A\|_{op} = \sup_{x\in S} \|A(x)\|$ where $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Prove the following: (i) $\|\cdot\|_{op}$ is a norm on $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ and that if $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ then $\|A\|_{op} = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|A(x)\|}{\|x\|}$; (ii) if $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ and $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^p)$ then $B \circ A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^p)$ and $\|B\circ A\|_{op} \leq \|B\|_{op}\|A\|_{op}$; (iii) if $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R})$ then there is $a \in \mathbb{R}^n$ such that $Ax = a \cdot x$ for all $x \in \mathbb{R}^n$ and in this case $\|A\|_{op} = \|a\|$; (iv) if $A \in \mathcal{L}(\mathbb{R};\mathbb{R}^m)$ then there is $a \in \mathbb{R}^m$ such that $Ax = xa$ for all $x \in \mathbb{R}$ and in this case $\|A\|_{op} = \|a\|$; (v) if $A \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ and (A_{ij}) is the matrix of A relative to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, then $\frac{1}{\sqrt{n}}\left(\sum_{j=1}^n\sum_{i=1}^m A_{ij}^2\right)^{1/2} \leq \|A\|_{op} \leq \left(\sum_{j=1}^n\sum_{i=1}^m A_{ij}^2\right)^{1/2}$, with equality on the right hand side inequality if and only if either $A = 0$ or $\operatorname{rank}(A) = 1$.

3. Let $\mathcal{M}_n$ be the space of $n \times n$ real matrices equipped with a norm. Show that the determinant function $\det\colon \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix $I \in \mathcal{M}_n$ with $D\det(I)(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det(A)(H) = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2\det(I)$ as a bilinear map.

4. Let $U \subset \mathbb{R}^n$ be open and path-connected, and let $f : U \to \mathbb{R}^m$ be differentiable. Recall from lectures that if $\|Df(x)\|_{op} \leq M$ for some constant M and all x in a ball

$B \subset U$, then $f|_B$ is Lipschitz on B (in fact $\|f(x) - f(y)\| \le M\|x - y\|$ for all $x, y \in B$.) If $\|Df(x)\|_{op} \le M$ for all $x \in U$, does it follow that f is Lipschitz on U?

5. Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ be a C^2 function and let $a \in U$.
(i) Recall that if a is a critical point of f and if the Hessian form $H_{f,a}(u) = \sum_{i,j=1}^n D_{ij} f(a) u_i u_j$ is positive definite (resp. negative definite), then f has a strict local minimum (resp. strict local maximum) at a. Prove that if a is a critical point of f and if $H_{f,a}$ is *indefinite* (i.e. if $\exists\, u_1, u_2 \in \mathbb{R}^n$ such that $H_{f,a}(u_1) > 0$ and $H_{f,a}(u_2) < 0$), then f has neither a local minimum nor a local maximum at a.
(ii) Prove that if f has a local minimum (resp. local maximum) at a then $Df(a) = 0$ and $H_{f,a}$ is positive semi-definite (resp. negative semi-definite).

6. (a) (*Necessity of continuity of second order partial derivatives in certain theorems*). Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Verify that: (i) f is C^1 on $\mathbb{R}^2$; (ii) all second order partial derivatives of f exist on $\mathbb{R}^2$; (iii) $\frac{\partial f}{\partial x \partial y}(0) \neq \frac{\partial f}{\partial y \partial x}(0)$; (iv) $Df(0) = 0$; (v) there are no constants $a, b, c \in \mathbb{R}$ such that $f(x, y) = ax^2 + bxy + cy^2 + E(x, y)$ for $E(x, y) = o(\|(x, y)\|^2)$.
(b) Find all critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \frac{1}{3}(x^3 + y^3) - x^2 - 2y^2 - 3x + 3y$, and for each critical point $a \in \mathbb{R}^2$, determine whether f has a local minimum, local maximum or neither at a.

7. Let U be a bounded open subset of $\mathbb{R}^n$ and let $f : \overline{U} \to \mathbb{R}$ be continuous on $\overline{U}$ (the closure of U in $\mathbb{R}^n$) and C^2 on U. Suppose that f satisfies the partial differential inequality $\Delta f(x) + \mathbf{b}(x) \cdot Df(x) + c(x)f(x) \ge 0$ for every $x \in U$ where Δ is the Laplace's operator defined by $\Delta f = \sum_{i=1}^n D_{ii} f$, and $\mathbf{b} : U \to \mathbb{R}^n$, $c : U \to \mathbb{R}$ are any functions with $c(x) < 0$ for each $x \in U$. If f is positive somewhere in $\overline{U}$, show that

$$\sup_{\overline{U}} f = \sup_{\partial U} f$$

where $\partial U = \overline{U} \setminus U$ is the boundary of U. [*Hint: if not, argue that f must have a positive local maximum at some (interior) point $x_0 \in U$, and consider $\Delta f(x_0) + \mathbf{b}(x_0) \cdot Df(x_0) + c(x_0)f(x_0)$*].

Deduce that if $\mathbf{b}$, c are as above, and if $\varphi : \partial U \to \mathbb{R}$ is a given continuous function, then for any $g : U \to \mathbb{R}$ there is at most one continuous function f on $\overline{U}$ that is C^2 on U and solves the boundary value problem $\Delta f + \mathbf{b} \cdot Df(x) + cf = g$ in U, $f = \varphi$ on ∂U.

8. Use the Contraction Mapping Theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using a calculator (remember to work in radians!), and justify the claimed accuracy of your approximation.

9. Give an example of a non-empty complete metric space (X, d) and a function $f : X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all x, $y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty compact subset of $\mathbb{R}^n$ with the

Euclidean metric. Show that in this case f must have a fixed point. If $g\colon X \to X$ satisfies $d(g(x), g(y)) \le d(x, y)$ for all x, $y \in X$, must g have a fixed point?

10. Let $\mathcal{M}_n$ be the space of $n \times n$ real matrices equipped with a norm, and let $f\colon \mathcal{M}_n \to \mathcal{M}_n$ be the map $f(A) = A^2$. Show that f is differentiable and that $Df\colon \mathcal{M}_n \to \mathcal{L}(\mathcal{M}_n; \mathcal{M}_n)$ is continuous on all of $\mathcal{M}_n$. Deduce that there is a continuous square-root function near the identity $I \in \mathcal{M}_n$; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a continuous function $g\colon B_\varepsilon(I) \to \mathcal{M}_n$ such that $(g(A))^2 = A$ for all $A \in B_\varepsilon(I)$. Is it possible to define a continuous square-root function on all of $\mathcal{M}_n$?

11. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Suppose that $\|Df(x) - I\|_{op} \le \mu$ for some $\mu \in (0, 1)$ and all $x \in \mathbb{R}^n$, where I is the identity map on $\mathbb{R}^n$ and $\|\cdot\|_{op}$ is the operator norm. Show that f is an open mapping, i.e. that f maps open subsets to open subsets. Show that $\|x - y\| \le (1 - \mu)^{-1}\|f(x) - f(y)\|$ for all $x, y \in \mathbb{R}^n$, and deduce that f is one-to-one and that $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$. Conclude that f is a diffeomorphism of $\mathbb{R}^n$, i.e. that f is a bijection with C^1 inverse. What can you say about a C^1 map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ assumed to satisfy only that $\|Df(x) - I\| < 1$ for all $x \in \mathbb{R}^n$?

12. Let $C = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Show that for each point $(x_0, y_0) \in C \setminus \{(0, 0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$, there exists an open set $U \subset \mathbb{R}^2$ containing (x_0, y_0), an open interval $I \subset \mathbb{R}$ containing x_0 and a C^1 function $g\colon I \to \mathbb{R}$ such that $C \cap U = \operatorname{graph} g \equiv \{(x, g(x)) : x \in I\}$.

13. Let $F\colon [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $x_0 \in \mathbb{R}^n$ and $R > 0$. Suppose that $\sup_{[a,b]\times\overline{B_R(x_0)}} \|F\| \le R(b - a)^{-1}$ and that $\|F(t, x) - F(t, y)\| \le K\|x - y\|$ for some K and all $t \in [a, b]$, $x, y \in \overline{B_R(x_0)}$. Recall from lectures that under these hypotheses, for each $t_0 \in [a, b]$, there is a unique $f \in C([a, b]; \overline{B_R(x_0)})$ solving the integral equation $f(t) = x_0 + \int_{t_0}^{t} F(s, f(s))\, ds$, $t \in [a, b]$. Show that this f is in fact the unique function in $C([a, b]; \mathbb{R}^n)$ solving the integral equation. [*Hint: for* $g \in C([a, b]; \mathbb{R}^n)$ *solving* $g(t) = x_0 + \int_{t_0}^{t} F(s, g(s))\, ds$, $t \in [a, b]$, *let* $\Lambda^+ = \{t \in [t_0, b] : \|g(\sigma) - x_0\| \le R \ \forall \sigma \in [t_0, t]\}$ *and consider the possibility that* $\sup \Lambda^+ < b$].