ANALYSIS II—EXAMPLES 3 Michaelmas 2025

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Assume that all vector spaces referred to below are over $\mathbb{R}$.

1. Quickies: (a) Show that two norms $\|\cdot\|$, $\|\cdot\|'$ on a vector space V are Lipschitz equivalent if and only if there exist numbers $r, R > 0$ such that $B_r \subseteq B'_1 \subseteq B_R$, where $B_\rho = \{x \in V : \|x\| < \rho\}$ and $B'_\rho = \{x \in V : \|x\|' < \rho\}$.

(b) Show that two norms $\|\cdot\|$, $\|\cdot\|'$ on a vector space V are Lipschitz equivalent if and only if the following holds: for any sequence (x_n) in V, $x_n \to x$ with respect to $\|\cdot\| \iff x_n \to x$ with respect to $\|\cdot\|'$.

(c) If $(V, \|\cdot\|)$ is a normed space and $\varphi : V \to \mathbb{R}$ is a linear functional, show that $\|\cdot\| + |\varphi(\cdot)|$ defines a norm on V, and that this norm is not Lipschitz equivalent to $\|\cdot\|$ if φ is not continuous.

(d)* Let $(V, \|\cdot\|)$ be a normed space. If any norm on V is Lipschitz equivalent to $\|\cdot\|$, does it follow that V is finite dimensional?

2. Let (X, d) and (X', d') be metric spaces. Prove that a map $f : X \to X'$ is continuous if and only if the inverse image $f^{-1}(V)$ of any open set $V \subset X'$ is open in X.

3. If X is a subset of $\mathbb{R}^n$ with the Euclidean metric and if every continuous function $f : X \to \mathbb{R}$ has bounded image, prove that X is compact. * Does this generalise to arbitrary metric spaces (X, d)?

4. (a) Let (X, d) be a totally bounded metric space (that is, (X, d) has the property that for every $\epsilon > 0$, there is a finite set $\{x_1, x_2, \ldots, x_N\} \subset X$ such that $X = \cup_{j=1}^N B_\epsilon(x_j)$). Show that any sequence (x_k) in X has a Cauchy subsequence. (b) Show that a metric space is compact if and only if it is complete and totally bounded.

5. Each of the following properties/notions makes sense for an arbitrary metric spaces X. Which are topological (i.e. dependent only on the collection of open subsets of X and not on the metric generating the open subsets)? Justify your answers.

(i) boundedness of a subset of X.

(ii) closed-ness of a subset of X.

(iii) notion that a subset of X is closed *and* bounded.

(iv) total boundedness of X.

(v) completeness of X.

(vi) notion that X is complete *and* totally bounded.

6. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. A differentiable curve passing through a is a differentiable map $\gamma : (-1, 1) \to \mathbb{R}^n$ with $\gamma(0) = a$. If $f \circ \gamma$ is differentiable at 0 for every differentiable curve γ passing through a, does it follow that f is differentiable at a?

7. Define $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (e^{x+y+z}, \cos x^2 y)$. Without making use of partial derivatives, show that f is everywhere differentiable and find $Df(a)$ at each $a \in \mathbb{R}^3$. Find all partial derivatives of f and hence, using appropriate results on partial derivatives, give an alternative proof of this result.

8. Consider the map $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df(x)(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

9. At which points of $\mathbb{R}^2$ is the function $f(x, y) = |x||y|$ differentiable? What about the function $g: \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) = xy/\sqrt{x^2 + y^2}$ if $(x, y) \neq (0, 0)$, $g(0, 0) = 0$?

10. Let f be a real-valued function on an open subset U of $\mathbb{R}^2$ such that that $f(\cdot, y)$ is continuous for each fixed $y \in U$ and $f(x, \cdot)$ is continuous for each fixed $x \in U$. Give an example to show that f need not be continuous on U. If additionally $f(\cdot, y)$ is Lipschitz for each $y \in U$ with Lipschitz constant independent of y, show that f is continuous on U. Deduce that if $D_1 f$ exists and is bounded on U and $f(x, \cdot)$ is continuous for each fixed $x \in U$, then f is continuous on U.

11. Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}^2$. If $D_1 f$ exists in some open ball around a and is continuous at a, and if $D_2 f$ exists at a, show that f is differentiable at a.

12.* Let $\mathcal{P}$ be the vector space of real polynomials on the unit interval $[0, 1]$. Show that for any infinite set $I \subseteq [0, 1]$, $\|p\|_I = \sup_I |p|$ defines a norm on $\mathcal{P}$. Use this fact to produce an example of a vector space, a sequence in it and two different norms on it such that the sequence converges to different elements in the space with respect to the different norms. (Hint: the Weierstrass approximation theorem may be helpful).

Is it possible to find such a sequence in one of the spaces ℓ^1 or ℓ^2 equipped with two norms, when possible, chosen from the standard norms on the spaces ℓ^1, ℓ^2, ℓ^∞? What about in the space $C([0, 1])$ equipped with two norms chosen from the L^1, L^2, L^∞ norms?

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* **Supplement: Incompleteness of the Riemann L^1 norm.** From lectures/ex. sheet 2, we know the following:

(i) $C([0, 1])$ with the norm $\|f\|_1 = \int_0^1 |f|$ is incomplete (ex. sheet 2, Q6);

(ii) $C([0, 1])$ is a linear subspace of the space of bounded Riemann integrable functions $\mathcal{R}([0, 1])$;

(iii) $\| \cdot \|_1$ extends to this larger space $\mathcal{R}([0, 1])$, although since

$$\int_0^1 |f| = 0 \not\Rightarrow f(x) = 0 \text{ for every } x \in [0, 1],$$

technically $\|\cdot\|_1$ is not a norm on $\mathcal{R}([0,1])$;

(iv) This issue is easily fixed by considering instead the quotient space $\tilde{\mathcal{R}}([0,1]) = \mathcal{R}([0,1])/\sim$ where $f \sim g$ if $f(x) = g(x)$ for a.e. $x \in [0,1]$ (i.e. there is a null set $N \subset [0,1]$ such that $f(x) = g(x)$ for every $x \in [0,1] \setminus N$). As discussed in lectures (as a non-examinable topic), addition $[f] + [g] = [f+g]$ and scalar multiplication $\lambda[f] = [\lambda f]$ are well-defined operations on $\tilde{\mathcal{R}}([0,1])$ which make it a vector space over $\mathbb{R}$, and $\|[f]\|_1 = \int_0^1 |f|$ is well-defined and is a norm on $\tilde{\mathcal{R}}([0,1])$ (this last assertion can be verified, for instance, with the help of Lebesgue's theorem on the Riemann integral, as discussed in lecture).

For notational simplicity we shall continue to write $\mathcal{R}([0,1])$ for $\tilde{\mathcal{R}}([0,1])$ and $\|f\|_1$ for $\|[f]\|_1$. Given the above facts, an important question is whether $\mathcal{R}([0,1])$ is complete with respect to $\|\cdot\|_1$.

To understand this, we might at first consider the following example which you saw very early on in the lectures: let $q_1, q_2, q_3, \ldots$ be an enumeration of the rational numbers in $[0,1]$, and let h_k be the indicator function of the set $\{q_1, q_2, \ldots, q_k\}$. We have that $h_k \to h$ pointwise on $[0,1]$ where $h : [0,1] \to \mathbb{R}$ is the indicator function of the set of all rationals in $[0,1]$. The h_k are Riemann integrable, but h is not; moreover, (h_k) is a Cauchy sequence with respect to $\|\cdot\|_1$.

Question: Does the sequence (h_k) have an L^1 limit? Why does this example not establish incompleteness of $(\mathcal{R}([0,1]), \|\cdot\|_1)$?

Here is an outline of a slightly more elaborate construction, which follows the basic idea of the above example but considers, instead of h_k, the indicator function f_k of a "fattened-up" version of $\{q_1, q_2, \ldots, q_k\}$. As an optional exercise, you may wish to complete this (to see incompleteness :)). Incompleteness of the L^1 norm on $\mathcal{R}([0,1])$ is a serious drawback of Riemann integration, which is very successfully resolved by the more general theory of Lebesgue integration (covered in Part II courses 'Probability & Measure' and 'Analysis of Functions').

Let $I_j = [q_j - \frac{1}{2^{j+2}}, q_j + \frac{1}{2^{j+2}}] \cap [0,1]$ and let $J_k = \cup_{j=1}^k I_j$. Let $f_k : [0,1] \to \mathbb{R}$ be the indicator function of J_k.

(a) Show that f_k is Riemann integrable, with $0 \le \int_0^1 f_k \le 1/2$ for each k.

(b) Show that (f_k) is a Cauchy sequence with respect to $\|\cdot\|_1$.

(c) Suppose (for a contradiction) that there is (a bounded) $g \in \mathcal{R}([0,1])$ such that $\|f_k - g\|_1 \to 0$. Show that $\int_{I_j} |1-g| = 0$ for each j.

(d) For any given interval $[p,q] \subset [0,1]$ with $p < q$, by choosing appropriate I_j and using part (c), show that $\sup_{[p,q]} g \ge 1$.

(e) Deduce that any Riemann upper sum $U(P,g) \ge 1$ and hence that $\int_0^1 g \ge 1$, which (since $\|f_k - g\|_1 \to 0$) contradicts part (a) . Conclude that $(\mathcal{R}([0,1]), \|\cdot\|_1)$ is incomplete.