ANALYSIS II—EXAMPLES 2 Michaelmas 2025

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1. Quickies: (a) If (X, d) is a metric space and A is a non-empty subset of X, show that the distance from $x \in X$ to A defined by $\rho(x) = \inf_{z \in A} d(x, z)$ is a Lipschitz function on X with Lipschitz constant ≤ 1, i.e. satisfies $|\rho(x) - \rho(y)| \leq d(x, y)$ for all $x, y \in X$.

(b) If (x_n), (y_n) are Cauchy sequences in a metric space (X, d), show that $(d(x_n, y_n))$ is convergent (in $\mathbb{R}$).

(c) Let (X, d) be a metric space, $x \in X$ and let (x_k) be a sequence in X such that $x_k \neq x$ for each k and $x_k \to x$. Given $y \in X$ with $y \neq x$, does there exist a metric d_1 on X such that $x_k \to y$ with respect to d_1?

(d) Use equivalence of norms on a finite dimensional vector space to show that for each n, there is a constant C such that the following holds: for every polynomial p of degree $\leq n$ there is $x_0 \in [0, 1/n]$ such that $|p(x)| \leq C|p(x_0)|$ for every $x \in [0, 1]$.

2. (a) Is the set $(1, 2]$ an open subset of the metric space $\mathbb{R}$ with the metric $d(x, y) = |x - y|$? Is it closed? What if we replace the ambient space $\mathbb{R}$ with the space $[0, 2]$, the space$(1, 3)$ or the space $(1, 2]$, in each case with the metric d?

(b) Let X be a set equipped with the discrete metric, and Y any metric space. Describe all open subsets of X, closed subsets of X, compact subsets of X, Cauchy sequences in X, continuous functions $f : X \to Y$ and continuous functions $f : Y \to X$.

3. For each of the following sets X, determine whether the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ small.

(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$; $d(x, x) = 0$, and, for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$, and, for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

4. (a) Let $C^1([0, 1])$ be the vector space of real continuous functions on $[0, 1]$ with continuous first derivatives. Define functions $\alpha, \beta, \gamma, \delta : C^1([0, 1]) \to \mathbb{R}$ by $\alpha(f) = \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)|$; $\beta(f) = \sup_{x \in [0,1]} (|f(x)| + |f'(x)|)$; $\gamma(f) = \sup_{x \in [0,1]} |f(x)|$; $\delta(f) = \sup_{x \in [0,1]} |f'(x)|$. Which of these define norms on $C^1([0, 1])$? Out of those that define norms, which pairs are Lipschitz equivalent?

(b) Let $C^1_c([0, 1])$ be the set of functions $f \in C^1([0, 1])$ such that $f(x) = 0$ for x in some neighborhood of the end points 0 and 1. Verify that $C^1_c([0, 1])$ is a vector space. How would your answers in (a) change if we replace $C^1([0, 1])$ by $C^1_c([0, 1])$?

5. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean metric are open? Which are closed? (And why?)

(i) $\{(x, 0) : 0 \leq x \leq 1\}$;

(ii) $\{(x, 0) : 0 < x < 1\}$;

(iii) $\{(x, y) : y \neq 0\}$;

(iv) $\{(x, y) : x \in \mathbb{Q} \ \text{ or } \ y \in \mathbb{Q}\}$;

(v) $\{(x, y) : y = nx \ \text{ for some } \ n \in \mathbb{N}\} \cup \{(x, y) : x = 0\}$;

(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

6. (a) Show that $\|f\|_1 = \int_0^1 |f|$ defines a norm on the vector space $C\big([0, 1]\big)$. Is it Lipschitz equivalent to the uniform norm $\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$? Is $C\big([0, 1]\big)$ with norm $\| \cdot \|_1$ complete?

(b) Let $\mathcal{R}([0, 1])$ denote the (real) vector space of all real-valued bounded Riemann integrable functions on $[0, 1]$. Is $\| \cdot \|_\infty$ a norm on $\mathcal{R}([0, 1])$? If so, is $\mathcal{R}([0, 1])$ with this norm complete?

(c) Why is $\| \cdot \|_1$ not, technically, a norm on $\mathcal{R}([0, 1])$? * What modification to $\mathcal{R}([0, 1])$ will fix the issue?

7. Is the set $\{f : f(1/2) = 0\}$ closed in the space $(C\big([0, 1]\big), \| \cdot \|_\infty)$? What about the set $\{f : \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform norm $\| \cdot \|_\infty$ with the norm $\|f\|_1 = \int_0^1 |f|$?

8. Let (X, d) be a metric space.

(a) Show that the union of any collection of open subsets of X must be open (regardless of whether the collection is finite, countable or uncountable), and that the intersection of any finite collection of open subsets is again open. Formulate and prove similar properties about closed subsets of X.

(b) Let E be a subset of X. The *interior* of E is the set $E^o = \{x \in X : B_\epsilon(x) \subset E \text{ for some } \epsilon > 0\}$ and the *closure* of E is the set $\overline{E} = \{x \in X : x_n \to x \text{ for some sequence } (x_n) \text{ in } E\}$. Show that E^o is the (unique) largest open subset of X contained in E, i.e. E^o is open in X, $E^o \subseteq E$, and if G is any open subset of X with $G \subseteq E$ then $G \subseteq E^o$. Show that $\overline{E}$ is the (unique) smallest closed subset of X containing E, i.e. $\overline{E}$ is closed in X, $E \subseteq \overline{E}$, and if F is any closed subset of X with $E \subseteq F$ then $\overline{E} \subseteq F$.

9. Let V be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $D_r(x) = \{y \in V : \|x - y\| \leq r\}$. Give an example to show that, in a general metric space (X, d), the closure of the open ball $B_r(x)$ need not be the closed ball $D_r(x) = \{y \in X : d(x, y) \leq r\}$.

10. Let $(x^{(n)})_{n\geq 1}$ be a bounded sequence in ℓ_∞, and write $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots)$. Show that there is a subsequence $(x^{(n_j)})_{j\geq 1}$ which converges in every coordinate; that is to say, the sequence $(x_i^{(n_j)})_{j\geq 1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ_∞ has a convergent subsequence?

11. Let ℓ_1 denote the set of real sequences (x_n) such that $\sum_{n=1}^\infty |x_n|$ is convergent. Show that, with addition and scalar multiplication defined termwise, ℓ_1 is a vector space. Define $\|\cdot\|_1 \colon \ell_1 \to \mathbb{R}$ by $\|x\|_1 = \sum_{n=1}^\infty |x_n|$. Show that $\|\cdot\|_1$ is a norm on ℓ_1, and that $(\ell_1, \|\cdot\|_1)$ is complete.

12. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)} \cdot y^{(m)}$ converges to $x \cdot y$. Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$, then so is the pointwise scalar product $f \cdot g : \mathbb{R}^n \to \mathbb{R}$.

13. For $f : [0, 1] \to \mathbb{R}^n$ a continuous function, write $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ for each $x \in [0, 1]$ and define $\int_0^1 f(x)\,dx = \left(\int_0^1 f_1(x)\,dx, \int_0^1 f_2(x)\,dx, \ldots, \int_0^1 f_n(x)\,dx\right)$.

(a) Let $v = \int_0^1 f(x)\,dx$. Show that $\|v\|_2^2 = \int_0^1 v \cdot f(x)\,dx$ where $\|\cdot\|_2$ is the Eucildean norm on $\mathbb{R}^n$. Deduce that $\|\int_0^1 f(x)\,dx\|_2 \leq \int_0^1 \|f(x)\|_2\,dx$.

(b)* Find all continuous $f : [0, 1] \to \mathbb{R}^n$ satisfying $\|\int_0^1 f(x)\,dx\| = \int_0^1 \|f(x)\|\,dx$ regardless of the norm $\|\cdot\|$.

14. Which of the following functions f are continuous?

(i) The linear map $f \colon \ell_\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$;

(ii) The identity map from the space $(C([0, 1]), \|\cdot\|_\infty)$ to the space $(C([0, 1]), \|\cdot\|_1)$;

(iii) The identity map from $(C([0, 1]), \|\cdot\|_1)$ to $(C([0, 1]), \|\cdot\|_\infty)$;

(iv) The linear map $f \colon \ell_0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$, where ℓ_0 has norm $\|\cdot\|_\infty$. (ℓ_0 is the space of real sequences (x_k) such that $x_k = 0$ for all but a finite number of k.)

15*. Let $(V, \|\cdot\|)$ be a normed space. Show that V is complete if and only if V has the property that for every sequence (x_n) in V with $\sum_{j=1}^\infty \|x_n\|$ convergent, the series $\sum_{n=1}^\infty x_n$ is convergent. (Thus V is complete if and only if every absolutely convergent series in V is convergent.) [Hint: If (x_n) is Cauchy, then there is a subsequence (x_{n_j}) such that $\sum_j \|x_{n_{j+1}} - x_{n_j}\| < \infty$.]

16. Let $(V, \|\cdot\|)$ be a normed space with the property that every bounded sequence in V has a convergent subsequence.

(a) Show that this property is equivalent to compactness of the unit sphere $S = \{x \in V : \|x\| = 1\}$.

(b) Show that $(V, \|\cdot\|)$ must be complete.

(c)* Show further that V must be finite-dimensional. [Hint: start by showing that for every finite-dimensional subspace V_0 of V, there exists $x \in V$ with $\|x + y\| > \|x\|/2$ for each $y \in V_0$.]

$\star$ **Supplement: Weierstrass approximation theorem.** This theorem says that *if* $f : [a, b] \to \mathbb{R}$ *is continuous, then there is a sequence of polynomials* (p_n) *such that* $p_n \to f$ *uniformly on* $[a, b]$. For those of you who are interested, here's an outline of an elementary proof that you can complete as an optional exercise. The proof is a very nice application of uniform continuity of continuous functions on closed, bounded intervals.

(a) By considering the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = f((1-x)a + xb)$, we may assume w.l.o.g. that $a = 0$, $b = 1$. For $n = 1, 2, 3, \ldots$, define *Bernstein polynomials*

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

We will see that these explicit polynomials (remarkably!) have the desired property, i.e. $p_n \to f$ uniformly on $[0, 1]$ as $n \to \infty$.

(b) First we need a few simple identities derived from the binomial expansion $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Show that $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$. By taking derivatives with respect to x and choosing y appropriately, show that $\sum_{k=1}^{n} k\binom{n}{k} x^{k-1}(1-x)^{n-k} = n$ and hence $\sum_{k=0}^{n} k\binom{n}{k} x^k (1-x)^{n-k} = nx$. By differentiating twice with respect to x, show that $\sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-2}(1-x)^{n-k} = n(n-1)$ and hence $\sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} = n(n-1)x^2$. Combining these deduce that

$$\sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x).$$

(c) Show that for any $x \in [0, 1]$, $p_n(x) - f(x) = \sum_{k=0}^{n} \binom{n}{k} (f(\frac{k}{n}) - f(x)) x^k (1-x)^{n-k}$.

(d) Let $\epsilon > 0$. Why can we find $\delta > 0$ such that $x, y \in [0, 1]$, $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$?

(e) Fix $x \in [0, 1]$ and $n \in \mathbb{N}$, and let $K = \{k \in \{0, 1, 2, \ldots, n\} : |x - \frac{k}{n}| < \delta\}$. Show that $|p_n(x) - f(x)| \leq \sum_{k=0}^{n} \binom{n}{k} |f(\frac{k}{n}) - f(x)| x^k (1-x)^{n-k} \leq \epsilon \sum_{\{k \in K\}} \binom{n}{k} x^k (1-x)^{n-k} + 2 \sup_{[0,1]} |f| \sum_{\{k \notin K\}} \binom{n}{k} x^k (1-x)^{n-k} \leq \epsilon \sum_{\{k \in K\}} \binom{n}{k} x^k (1-x)^{n-k} + \frac{2 \sup_{[0,1]} |f|}{n^2 \delta^2} \sum_{\{k \notin K\}} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k}$.

(f) Use (e) and the results of (b) to conclude that $\sup_{x \in [0,1]} |p_n(x) - f(x)| \leq \epsilon + \frac{2 \sup_{[0,1]} |f|}{n \delta^2}$, and hence that for all sufficiently large n, $\sup_{x \in [0,1]} |p_n(x) - f(x)| \leq 2\epsilon$. This proves the theorem.