ANALYSIS II—EXAMPLES 1 Michaelmas 2025

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1. Quickies: (a) If (f_n) is a sequence of real functions on the interval $[0,1]$ converging uniformly to a function f on $[0,1]$, and if f_n is continuous at $x_n \in [0,1]$ with $x_n \to x$, does it follow that f is continuous at x?

(b) If (f_n) is a sequence of continuous real functions on the interval $[-1,1]$ converging pointwise to a continuous function f on $[-1,1]$, and if the convergence is uniform on $[-r,r]$ for every $r \in (0,1)$, does it follow that the convergence is uniform on $[-1,1]$?

(c) If (f_n) is a sequence of real functions on the interval $[0,1]$ converging uniformly to a function f on $[0,1]$, and if each f_n is continuous except at countably many points, does it follow that there exists a point at which f is continuous?

(d)* If (f_n) is a sequence of differentiable functions on $[0,1]$ converging uniformly to a function f on $[0,1]$, does it follow that there exists a point at which f is differentiable?

2. Which of the following sequences (f_n) of functions converge uniformly on the set X?

(a) $f_n(x) = x^n$ on $X = (0,1)$; (b) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$; (c) $f_n(x) = xe^{-nx}$ on $X = [0,\infty)$;

(d) $f_n(x) = e^{-x^2}\sin(x/n)$ on $X = \mathbb{R}$.

3. Let (f_n) and (g_n) be sequences of real-valued functions on a set E converging uniformly to f and g respectively. Show that the sequence of pointwise sums $(f_n + g_n)$ converges uniformly to $f+g$. On the other hand, show that the sequence of pointwise products $(f_n g_n)$ need not converge uniformly to fg, but if both f and g are bounded then $(f_n g_n)$ does converge uniformly to fg. What if f is bounded but g is not?

4. Let (f_n) be a sequence of bounded, real-valued functions on a set E converging uniformly to a function f. Show that f must be bounded. Give an example of a sequence (g_n) of bounded, real-valued functions on $[-1,1]$ converging pointwise to a function g which is not bounded.

5. Let (f_n) be a sequence of real-valued continuous functions on a closed, bounded interval $[a,b]$, and suppose that f_n converges pointwise to a continuous function f. Show that if $f_n \to f$ uniformly and (x_n) is a sequence of points in $[a,b]$ with $x_n \to x$ then $f_n(x_n) \to f(x)$. On the other hand, show that if f_n does not converge uniformly to f then we can find a convergent sequence $x_n \to x$ in $[a,b]$ such that $f_n(x_n) \not\to f(x)$.

6. Let (f_n) be a sequence of real-valued functions on $[0,1]$ converging uniformly to a function f.

(a) If $\mathcal{D}_n$ is the set of discontinuities of f_n and $\mathcal{D}$ is the set of discontinuities of f, show that $\mathcal{D} \subseteq \cup_{n=1}^\infty \cap_{j=n}^\infty \mathcal{D}_j$.

(b) Suppose that for some finite k, each f_n is discontinuous at most at k points. What can you say about the set of discontinuities of f?

7. Let $\sum_{n=1}^\infty a_n$ be an absolutely convergent series of real numbers.

(a) Define a sequence (f_n) of functions on $[-\pi,\pi]$ by $f_n(x) = \sum_{m=1}^n a_m \sin mx$. Show that each f_n is differentiable with $f_n'(x) = \sum_{m=1}^n m a_m \cos mx$.

(b) Show that $f(x) = \sum_{m=1}^\infty a_m \sin mx$ defines a continuous function on $[-\pi,\pi]$, but that the series $\sum_{m=1}^\infty m a_m \cos mx$ need not converge.

8. Show that, for any $x \in X = \mathbb{R} \setminus \{1,2,3,\dots\}$, the series $\sum_{m=1}^\infty (x-m)^{-2}$ converges. Define $f : X \to \mathbb{R}$ by $f(x) = \sum_{m=1}^\infty (x-m)^{-2}$, and for $n = 1, 2, 3, \dots$, define $f_n : X \to \mathbb{R}$ by $f_n(x) = \sum_{m=1}^n (x-m)^{-2}$. Does the sequence (f_n) converge uniformly to f on X? Is f continuous?

9. Let a_n be real numbers such that $\sum_{n=0}^{\infty} a_n$ converges.

(a) Show that $\sum_{n=1}^{\infty} a_n x^n$ converges for $x \in (-1,1)$. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, show that f is differentiable on $(-1,1)$.

(b)* Show that f extends to $(-1,1]$ as a continuous function with $f(1) = \sum_{n=0}^{\infty} a_n$. (Hint: start by showing that $f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$ for $|x| < 1$, where $s_n = \sum_{j=0}^{n} a_j$.) Show that, for each $r \in (-1,1)$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[r,1]$. Must the one-sided derivative $f'(1)$ exist?

10. Is there a real power series with radius of convergence 1 that converges uniformly on $(-1,1)$?

11. Which of the following functions $f\colon [0,\infty) \to \mathbb{R}$ are (a) uniformly continuous; (b) bounded?

(i)$f(x) = \sin x^2$; (ii) $f(x) = \inf\{|x-n^2| : n \in \mathbb{N}\}$; (iii) $f(x) = (\sin x^3)/(x+1)$.

12. Show that if (f_n) is a sequence of uniformly continuous, real-valued functions on $\mathbb{R}$, and if $f_n \to f$ uniformly, then f is uniformly continuous. Give an example of a sequence of uniformly continuous, real-valued functions (f_n) on $\mathbb{R}$ such that f_n converges pointwise to a function f which is continuous but not uniformly continuous.

13. Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous, and that $f(x)$ tends to a finite limit as $x \to \infty$. Must f be uniformly continuous on $[0,\infty)$? Give a proof or a counterexample as appropriate.

14. Let f be a differentiable, real-valued function on $\mathbb{R}$, and suppose that f' is bounded. Show that f is uniformly continuous. Let $g\colon [-1,1] \to \mathbb{R}$ be the function defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable, but that its derivative is unbounded. Is g uniformly continuous?

15. Let f be a bounded real-valued Riemann integrable functions on $[0,1]$.

(a) Must there exist a sequence (f_n) of continuous functions on $[0,1]$ such that $f_n \to f$ uniformly on $[0,1]$?

(b)* Must there exist a sequence (f_n) of continuous functions on $[0,1]$ such that $\int_0^1 |f_n(x) - f(x)|\,dx \to 0$?

(c)* Must there exist a sequence (p_n) of polynomials such that $\int_0^1 |p_n(x) - f(x)|\,dx \to 0$?

16*. (A continuous nowhere differentiable function). Define $\varphi(x) = |x|$ for $x \in [-1,1]$ and extend φ to $\mathbb{R}$ by requiring that $\varphi(x+2) = \varphi(x)$.

(i) Show that $|\varphi(s) - \varphi(t)| \leq |s-t|$ for all s and t.

(ii) Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Prove that f is well-defined (i.e. the series converges for each x) and that f is continuous.

(iii) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2}4^{-m}$, where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. Prove that

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \geq \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x. Hence there exists a real continuous function on the real line which is nowhere differentiable.