Analysis II **Michaelmas 2017**

Example Sheet 1

1. Prove the following facts about convergence of sequences in a normed space:
   (a) If $(\mathbf{v}_n) \to \mathbf{v}$ and $(\mathbf{w}_n) \to \mathbf{w}$, then $(\mathbf{v}_n + \mathbf{w}_n) \to \mathbf{v} + \mathbf{w}$.
   (b) If $(\mathbf{v}_n) \to \mathbf{v}$ and $\lambda \in \mathbb{R}$, then $(\lambda \mathbf{v}_n) \to \lambda \mathbf{v}$.
   (c) If $(\mathbf{v}_n) \to \mathbf{v}$, then any subsequence $(\mathbf{v}_{n_i})$ of $(\mathbf{v}_n)$ also converges to $\mathbf{v}$.
   (d) If $(\mathbf{v}_n) \to \mathbf{v}$ and $\mathbf{v}_n \to \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

   Using (a) and (b), show that if $f, g : V \to W$ are continuous, so is $f + \lambda g$, where $\lambda \in \mathbb{R}$.

2. Suppose X is a finite subset of $\mathbb{R}^n$ whose elements span $\mathbb{R}^n$. Show that

$$\|\mathbf{v}\|_X = \max_{\mathbf{w} \in X} |\mathbf{v} \cdot \mathbf{w}|$$

   defines a norm on $\mathbb{R}^n$. Find a norm on $\mathbb{R}^2$ whose closed unit ball is a regular (Euclidean) hexagon.

3. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? Why?
   (a) $\{(x, 0) \mid 0 \leq x \leq 1\}$
   (b) $\{(x, 0) \mid 0 < x < 1\}$
   (c) $\{(x, y) \mid y \neq 0\}$
   (d) $\{(x, y) \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$
   (e) $\{(x, y) \mid y = nx \text{ for some } n \in \mathbb{N}\}$

4. Is the set $\{f \in C[0,1] \mid f(1/2) = 0\}$ a closed subset of $C[0,1]$ with respect to $\|\cdot\|_\infty$? With respect to $\|\cdot\|_1$? What about the set $\{f \in C[0,1] \mid \int_0^1 f(x)\,dx = 0\}$?

5. Let ℓ_0 be the set of real sequences (x_n) such that all but finitely many x_n are 0. If we use the natural definition of addition and scalar multiplication: $(x_n) + (y_n) = (x_n + y_n)$ and $\lambda(x_n) = (\lambda x_n)$, then ℓ_0 is a vector space. Find two norms on ℓ_0 which are not Lipshitz equivalent. Can you find uncountably many?

6. Suppose V and W are normed spaces, and that $L : V \to W$ is a linear map. Show that L is continuous if and only if the set $S(L) = \{\|L\mathbf{v}\|/\|\mathbf{v}\| \mid \mathbf{v} \in V \setminus \mathbf{0}\}$ is bounded above. Let $\mathcal{B}(V, W) = \{L : V \to W \mid L \text{ is linear and continuous}\}$. For $L \in \mathcal{B}(V, W)$, let $\|L\| = \sup S(L)$.

   (a) Show that $\|\cdot\|$ defines a norm on $\mathcal{B}(V, W)$. (This is called the *operator norm.*)

   (b) If $L_1 \in \mathcal{B}(V_1, V_2)$ and $L_2 \in \mathcal{B}(V_2, V_3)$, show that $\|L_2 \circ L_1\| \leq \|L_2\|\|L_1\|$.

   (c) Now suppose $V = W = \mathbb{R}^n$ with the Euclidean norm, and that L is given by multiplication by a symmetric matrix A. What is $\|L\|$?

7. Which of the following sequences of functions (f_n) converge uniformly on the set X?

   (a) $f_n(x) = x^n$ on $X = (0, 1)$

(b) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$
(c) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$
(d) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

8. Consider the functions $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = n^p x \exp(-n^q x)$, where p and q are positive constants.
   (a) Show that (f_n) converges pointwise on $[0,1]$ for any values of p and q.
   (b) Show that if $p < q$, then (f_n) converges uniformly on $[0,1]$.
   (c) Show that if $p \geq q$, then (f_n) does not converge uniformly on $[0,1]$.

9. Let $f_n(x) = n^\alpha x^n (1-x)$, where α is a real constant.
   (a) For which values of α does $f_n(x) \to 0$ pointwise on $[0,1]$?
   (b) For which values of α does $(f_n) \to 0$ uniformly on $[0,1]$?
   (c) For which values of α does $(f_n) \to 0$ with respect to $\|\cdot\|_1$?
   (d) For which values of α does $f_n'(x) \to 0$ pointwise on $[0,1]$?
   (e) For which values of α does $(f_n') \to 0$ uniformly on $[0,1]$?

10. Let $\sum_{n=1}^\infty a_n$ be an absolutely convergent series of real numbers. Show that $f(x) = \sum_{n=1}^\infty a_n \sin nx$ defines a continuous function on $\mathbb{R}$, but that the series $\sum_{n=1}^\infty n a_n \cos nx$ need not converge.

11. Consider the sequence of functions $f_n : (\mathbb{R} - \mathbb{Z}) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{m=0}^{n} (x-m)^{-2}.$$

Show that (f_n) converges pointwise on $\mathbb{R} - \mathbb{Z}$ to a function f. Does (f_n) converge uniformly to f? Is f continuous on $\mathbb{R} - \mathbb{Z}$?

12. * If a_n are real numbers such that $\sum_{n=0}^\infty a_n$ converges, show that $\sum_{n=0}^\infty a_n x^n$ converges for $x \in (-1,1)$. If $f(x) = \sum_{n=0}^\infty a_n x^n$, show that f extends to a continuous function on $(-1,1]$ with $f(1) = \sum_{n=0}^\infty a_n$. (*Hint:* show that for $x \in (-1,1)$, $f(x) = (1-x)\sum_{n=0}^\infty s_n x^n$, where $s_n = \sum_{j=0}^n a_j$.) Show that for each $r \in (-1,1)$, the series $\sum_{n=0}^\infty a_n x^n$ converges uniformly on $[r,1]$. Must the one-sided derivative $f'(1)$ exist?

13. * Define $\varphi(x) = |x|$ for $x \in [-1,1]$ and extend the definition of $\varphi(x)$ to all of $\mathbb{R}$ by requiring that $\varphi(x+2) = \varphi(x)$.
   (a) Show that $|\varphi(s) - \varphi(t)| \leq |s-t|$ for all $s, t \in \mathbb{R}$.
   (b) Define $f(x) = \sum_{n=0}^\infty \left(\frac{3}{4}\right)^n \phi(4^n x)$. Prove that f is well-defined and continuous.
   (c) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2}4^{-m}$, where the sign is chosen so that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. Show that

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \geq \frac{1}{2}(3^m + 1).$$

   Deduce that $f : \mathbb{R} \to \mathbb{R}$ is continuous but nowhere differentiable.

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