Mich. 2013 **ANALYSIS II—EXAMPLES 4** PAR

1. (a) Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and let $f_1, \ldots, f_m$ be the coordinate functions of f; that is to say, each $f_i: \mathbb{R} \to \mathbb{R}$ and, for all $x \in \mathbb{R}^n$, $f(x) = (f_1(x) \ldots, f_m(x))$. Show that f is differentiable at $x \in \mathbb{R}_n$ iff each f_i is, and that $Df|_x(h) = (Df_1|_x(h), \ldots, Df_m|_x(h))$.

(b) Define $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (e^{x+y+z}, \cos x^2 y)$. Without making use of partial derivatives, show that f is everywhere differentiable and find its derivative at each point $(x, y, z) \in \mathbb{R}^3$.

(c) Find the matrix of partial derivatives of f and hence, using appropriate results on partial derivatives, give an alternative proof of the result of (b).

2. Define $f: \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^4$. Show that f is differentiable at every $A \in \mathcal{M}_n$, and find $Df|_A$ as a linear map. Show further that f is twice-differentiable at every $A \in \mathcal{M}_n$ and find $D^2f|_A$ as a bilinear map from $\mathcal{M}_n \times \mathcal{M}_n$ to $\mathcal{M}_n$.

3. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

4. Consider the map $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df|_x(h)$ is orthogonal to x and explain geometrically why this is the case.

5. At which points is the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |x||y|$ differentiable? What about the function $g: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} xy/\sqrt{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}?$$

6. Show that the function $\det: \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D \det|_I(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D \det|_A(H) = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2 \det|_I$ as a bilinear map.

7. Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $(a, b) \in \mathbb{R}^2$.

(a) Suppose that $D_1 f$ exists and is continuous in some open ball around (a, b), and that $D_2 f$ exists at (a, b). Show that f is differentiable at (a, b).

(b) Suppose instead that $D_1 f$ exists and is bounded on some open ball around (a, b), and that for fixed x the function $y \mapsto f(x, y)$ is continuous. Show that f is continuous at (a, b).

8. Define $f: \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on the whole of $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a continuous function $g: B_\varepsilon(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

9. Consider the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x, x^3 + y^3 - 3xy)$ and the set $C = \{(x, y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Show that f is locally invertible around each point of C except $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0, y_0) \in C \backslash \{(0, 0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $f(x_0, y_0)$ such that f maps U bijectively to V. What is the derivative of the local inverse function? Deduce that for each point $(x_0, y_0) \in C$ other than $(0, 0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$ there exist open intervals I containing x_0 and J containing y_0 such that for each $x \in I$ there is a unique $y \in J$ with $(x, y) \in C$.

10. Use the contraction mapping theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using an electronic pocket calculator, and justify the claimed accuracy of your approximation.

11. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x \in I} |f(x) e^{-\alpha x}|$, and that the norm $\| \cdot \|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x \in I} |f(x)|$.

Now suppose that $\phi: \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable. Consider the map $T: C(I) \to C(I)$ defined by $T(f)(x) = y_0 + \int_0^x \phi(t, f(t)) dt$. Give an example to show that T need not be a contraction under the uniform norm. Show, however, that T is a contraction under the norm $\| \cdot \|_\alpha$ for some α, and hence deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

12. Let (X, d) be a non-empty complete metric space, let $f: X \to X$ be a continuous function, and let $K \in [0, 1)$.

(a) Suppose we assume that for all x, $y \in X$ we have either $d(f(x), f(y)) \leq K d(x, y)$ or $d(f(f(x)), f(f(y))) \leq K d(x, y)$. Show that f has a fixed point.

$^+$(b) Suppose instead we assume only that for all x, $y \in X$ at least one of the three distances $d(f(x), f(y))$, $d(f(f(x)), f(f(y)))$ and $d(f(f(f(x))), f(f(f(y))))$ is less than or equal to $K d(x, y)$. Must f have a fixed point?