MATHEMATICAL TRIPOS: PART IA **Lent 2024**

PROBABILITY **JRN**

Example Sheet 4 (of 4)

1. Alice and Bob agree to meet in the Copper Kettle after their Saturday lectures. They arrive at times that are independent and uniformly distributed between 12.00 and 1.00 pm. Each is prepared to wait 10 minutes before leaving. Find the probability they meet.

2. A stick is broken in two places, independently uniformly distributed along its length. What is the probability that the three pieces will make a triangle?

3. The radius of a circle is exponentially distributed with parameter λ. Determine the probability density function of the area of the circle.

4. The random variables X and Y are independent and exponentially distributed with parameters λ and μ respectively. Find the distribution of $\min\{X, Y\}$ and the probability that X exceeds Y.

5. How large a random sample should be taken from a normal distribution in order for the probability to be at least 0.99 that the sample mean will be within one standard deviation of the mean of the distribution? [*Hint. For the distribution function* Φ *of* $N(0,1)$ *we have* $\Phi(2.58) = 0.995$.]

6. A random variable X is said to have *log-normal distribution* if $Y = \log X$ is normally distributed.

(a) Find the mean and variance of X in the case where $Y \sim N(\mu, \sigma^2)$.

(b) Log-normal distributions are use used model quantities X which are believed to arise as the product of many positive random factors $X = \xi_1 \xi_2 \dots \xi_n$, such as particle sizes after a crushing process or stock prices. Making any reasonable assumptions you wish, give a justification for such a model.

7. Suppose that X and Y are independent $N(0,1)$ random variables. Show that, for any fixed θ, the random variables

$$U = X\cos\theta + Y\sin\theta \quad V = -X\sin\theta + Y\cos\theta$$

are independent and find their distributions.

8. The random variables X and Y are independent and exponentially distributed, each with parameter λ. Show that the random variables $X+Y$ and $X/(X+Y)$ are independent and find their distributions.

9. A shot is fired at a circular target. The vertical and horizontal coordinates of the bullet hole, with the centre of the target as origin, are taken to be independent $N(0,1)$ random variables.

(a) Show that the distance of the hole from the centre has density function $re^{-r^2/2}$ on $[0, \infty)$.

(b) Show that the mean of this distance is $\sqrt{\pi/2}$, the median is $\sqrt{\log 4}$, and the mode is 1.

10. A radioactive source emits particles in a random direction (with all directions being equally likely). It is held at a distance a from a vertical infinite plane photographic plate.

(a) Show that, given the particle hits the plate, the horizontal coordinate of its point of impact (with the point nearest the source as origin) has the Cauchy density function $a/(\pi(a^2 + x^2))$.

(b) Can you compute the mean of this distribution?

11. A random sample is taken in order to find the proportion of Labour voters in a population. Guided by the central limit theorem, determine a sample size such that the probability of a sampling error less than 0.04 will be 0.99 or greater.

12. Let $X_1, \dots, X_n$ be independent random variables with $\mathbb{E}(X_i) = \mu_i$ and $\mathrm{var}(X_i) = \sigma^2 < \infty$ for all i. Let a_i and b_i be real constants for all i and set

$$Y_1 = \sum_{i=1}^{n} a_i X_i, \quad Y_2 = \sum_{i=1}^{n} b_i X_i.$$

(a) Show that $\mathrm{cov}(Y_1, Y_2) = \sigma^2 \sum_{i=1}^{n} a_i b_i$.

(b) Prove that, if $X_1, \dots, X_n$ are independent normal random variables, then Y_1, Y_2 are independent if and only if $\mathrm{cov}(Y_1, Y_2) = 0$.

13. Show that, as $n \to \infty$,

$$e^{-n} \left(1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2}.$$