MATHEMATICAL TRIPOS: PART IA **Lent 2024**

PROBABILITY **JRN**

Example Sheet 2 (of 4)

1. A coin with probability $p \in [0,1]$ of heads is tossed n times. Let E be the event 'a head is obtained on the first toss' and F_k the event 'exactly k heads are obtained'. For which pairs of non-negative integers (n,k) are E and F_k independent?

2. The events A and B are independent. Show that the events A^c and B are independent, and that the events A^c and B^c are independent.

3. Independent trials are performed, each with probability p of success. Let P_n be the probability that n trials result in an even number of successes. Show that

$$P_n = \tfrac{1}{2}(1 + (1-2p)^n).$$

4. Two darts players A and B throw alternately at a board and the first to score a bull wins the contest. The outcomes of different throws are independent and on each of their throws A has probability p_A and B has probability p_B of scoring a bull. If A has first throw, calculate the probability p that A wins the contest.

5. Consider the probability space $\Omega = \{0,1\}^3$ with equally likely outcomes.

(a) Show that there are 70 different Bernoulli random variables of parameter $1/2$ that can be defined on Ω.

(b) How many Bernoulli random variables of parameter $1/3$ can be defined on Ω?

(c) What is the length of the longest sequence of independent Bernoulli random variables of parameter $1/2$ that can be defined on Ω?

6. Suppose that X and Y are independent Poisson random variables with parameters λ and μ respectively. Find the distribution of $X+Y$. Prove that the conditional distribution of X, given that $X+Y=n$, is binomial with parameters n and $\lambda/(\lambda+\mu)$.

7. The number of misprints on a page has a Poisson distribution with parameter λ, and the numbers on different pages are independent.

(a) What is the probability that the second misprint will occur on page r?

(b) A proof-reader studies a single page looking for misprints. She catches each misprint (independently of others) with probability $p \in [0,1]$. Let X be the number of misprints she catches and let Y be the number she misses. Find the distributions of the random variables X and Y and show they are independent.

8. Let $X_1, \dots, X_n$ be independent identically distributed random variables with mean μ and variance σ^2. Find the means of the random variables

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2.$$

9. In a sequence of n independent trials the probability of a success at the ith trial is p_i. Let N denote the total number of successes. Find the mean and variance of N.

10. Liam's bowl of spaghetti contains n strands. He selects two ends at random and joins them together. He repeats this until no ends are left. What is the expected number of spaghetti hoops in the bowl?

11. Sarah collects figures from cornflakes packets. Each packet contains one of n distinct figures. Each type of figure is equally likely. Show that the expected number of packets Sarah needs to buy to collect a complete set of n is

$$n\sum_{i=1}^{n}\frac{1}{i}.$$

12. Let $a_1, a_2, \dots, a_n$ be a ranking of the yearly rainfalls in Cambridge over the next n years. Assume that $a_1, a_2, \dots, a_n$ is a random permutation of $1, 2, \dots, n$. Say that k is a record year if $a_k < a_i$ for all $i < k$. Thus the first year is always a record year. Let $Y_i = 1$ if i is a record year and 0 otherwise. Find the distribution of Y_i and show that $Y_1, Y_2, \dots, Y_n$ are independent. Calculate the mean and variance of the number N of record years in the next n years.

13. Let $s \in (1, \infty)$ and let X be a random variable in $\{1, 2, \dots\}$ with distribution

$$\mathbb{P}(X = n) = n^{-s}/\zeta(s)$$

where $\zeta(s)$ is a suitable normalizing constant. For each prime number p let A_p be the event that X is divisible by p. Find $\mathbb{P}(A_p)$ and show that the events $(A_p : p \text{ prime})$ are independent. Deduce that

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}.$$