MATHEMATICAL TRIPOS: PART IA **Lent 2022**

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Example Sheet 1 (of 4)

Exercises

1. Four mice are chosen (without replacement) from a litter, two of which are white. The probability that both white mice are chosen is twice the probability that neither is chosen. How many mice are there in the litter?

2. A table-tennis championship for 2^n players is organized as a knock-out tournament with n rounds, the last round being the final. After the tournament is over, two players are chosen uniformly at random to be interviewed together. Calculate the probability that they had met: (a) in the first round, (b) in the final, (c) in any round.

[*Hint. The same probability space can be used for all three calculations.*]

3. A full deck of 52 cards is divided into two piles of 26 cards each, uniformly at random. Find an expression for the probability that each pile contains 13 red and 13 black cards. Evaluate this expression as a decimal expansion using a calculator or similar. Use Stirling's formula to find an approximation for the same probability and evaluate this approximation as a decimal expansion.

4. Remind yourself of what it means for $\mathcal{F}$ to be a σ-algebra and for $\mathbb{P}$ to be a probability measure. Let $(A_n)_{n\geq 1}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show, starting from the definitions, the following properties:

(a) $\emptyset$ and $A_1 \cup A_2$ and $\cap_{n=1}^{\infty} A_n$ are events,

(b) $\mathbb{P}(\emptyset) = 0$ and (as in lectures) $\mathbb{P}(A_1^c) = 1 - \mathbb{P}(A_1)$,

(c) if A_1 and A_2 are disjoint, then $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$,

(d) if $A_1 \subseteq A_2$, then $\mathbb{P}(A_1) \leq \mathbb{P}(A_2)$,

(e) $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$,

(f) if $A_n \subseteq A_{n+1}$ for all n, then $\mathbb{P}(A_n) \to \mathbb{P}(\cup_n A_n)$.

5. (a) Show that, for any three events A, B, C,

$$\mathbb{P}(A^c \cap (B \cup C)) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C).$$

(b) How many of the numbers $1, \ldots, 500$ are not divisible by 7 but are divisible by 3 or 5?

6. Let $(A_n : n \in \mathbb{N})$ be a sequence of events in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Set

$$A = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}, \quad B = \{\omega \in \Omega : \omega \in A_n \text{ for all sufficiently large } n\}.$$

(a) Show that $B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$.

(b) Show that A is an event and that $\mathbb{P}(A) \leq \sum_{k=n}^{\infty} \mathbb{P}(A_k)$ for all n.

(c) Suppose that the series $\sum_{n=1}^{\infty} \mathbb{P}(A_n)$ converges. Show that $\mathbb{P}(A) = 0$.

[*We often describe the events A, B as "(A_n) holds* <u>*infinitely often*</u>*" and "(A_n) holds* <u>*eventually*</u>*".*]

7. A committee of size r is chosen at random from a set of n people. Calculate the probability that m given people will all be on the committee (a) directly, (b) using the inclusion-exclusion formula. Deduce that

$$\binom{n-m}{r-m} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n-j}{r}.$$

8. Examination candidates are graded into four classes known conventionally as I, II-1, II-2 and III, with probabilities 1/8, 2/8, 3/8 and 2/8 respectively. Candidates who misread the rubric, a common event with probability 2/3, generally do worse, their probabilities being 1/10, 2/10, 4/10 and 3/10. What is the probability:

(a) that a candidate who reads the rubric correctly is placed in the class II-1?

(b) that a candidate who is placed in the class II-1 has read the rubric correctly?

9. A fictional college committee contains a proportion p of dons, who never change their minds about anything, and a proportion $1-p$ of student reps who change their minds completely at random (with probability r) between successive votes on the same issue. A randomly chosen committee member is noticed to have voted twice in succession in the same way. What is the probability that this member will vote in the same way next time?

10. Suppose that n balls are tossed independently and uniformly at random into n boxes. What is the probability that exactly one box is empty? Check your answer for $n = 2$ and $n = 3$ directly.

11. What is the probability that an increasing function $\{1, \ldots, k\} \to \{1, \ldots, n\}$ chosen uniformly at random is strictly increasing?

[*Here, a function f is increasing if $f(x) \leq f(y)$ whenever $x < y$, and strictly increasing if $f(x) < f(y)$ whenever $x < y$.*]

12. Let $(X_n)_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$, starting from 0.

(a) Show that, for $h = h(n) = 2/\sqrt{n}$, in the limit $n \to \infty$ with n even,

$$\mathbb{P}(X_n = 0) \sim \frac{1}{\sqrt{2\pi}} h.$$

(b) Show further that, for all $x \in \mathbb{R}$,

$$\mathbb{P}(X_n/\sqrt{n} \in [x, x+h)) \sim \frac{1}{\sqrt{2\pi}} h e^{-x^2/2}.$$

[*Hints. For all x and all n, $X_n/\sqrt{n}$ takes exactly one value in $[x, x+h)$. Recall that $(1+1/x)^x \to e$ as $x \to \pm\infty$.*]

Extensions

Some of the following questions have a more puzzle-like quality. Others are intended to be more open-ended, or general food for thought.

13. Go back to Exercise 3. Suppose instead we tossed a coin 26 times, and evaluated the probabiliity that we see 13 heads and 13 tails. *Without explicit calculation*, would you expect this new probability to be larger, smaller, or equal to the probability obtained in Exercise 3?

14. Go back to Exercise 2. Suppose instead that the players were chosen uniformly at random to be interviewed *before* the tournament started. Would the answers be the same? What care must we take if we are told that the results of the tournament are also determined randomly?

15. Mary tosses two coins and John tosses one coin. What is the probability that Mary gets more heads than John? What about if Mary tosses three coins and John tosses two. Make a conjecture for the probability when Mary tosses $n+1$ and John tosses n. Can you prove your conjecture?

16. You are hoping to make a uniform random choice from $\{1, 2, \dots, k\}$. Regrettably, all you have is an m-sided dice, for $m \neq k$. How would you proceed (a) if $m > k$? (b) if $m < k$?

17. Go back to Exercise 9. Imagine that p is not known, but $r = 1/2$. A *Varsity* investigative article claims that $p = 0.01$. You are unconvinced. You pick a member uniformly at random and observe their voting. They vote the same for 20 consecutive votes. Discuss whether you might obtain useful further information to assess the claim by watching them for the next 20 votes.

What about if *The Cambridge Student* had responded, claiming that $p = 0$?