MATHEMATICAL TRIPOS: PART IA **Lent 2018**

PROBABILITY **PS**

Example Sheet 3 (of 4)

1. Let $x_1, x_2, \dots, x_n$ be positive real numbers.

(a) Show that their harmonic mean is no greater than their arithmetic mean, that is,

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right)^{-1} \le \frac{1}{n}\sum_{i=1}^{n} x_i.$$

(b) Show that, if $y_1, \dots, y_n$ is any reordering of $x_1, \dots, x_n$, then

$$\frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i} \ge 1.$$

2. Let X be a random variable.

(a) Show that, for all $p \in (0, \infty)$ and all $x \in (0, \infty)$,

$$\mathbb{P}(|X| \ge x) \le \mathbb{E}(|X|^p)x^{-p}.$$

(b) Show that, for all $\beta \ge 0$,

$$\mathbb{P}(X \ge x) \le \mathbb{E}(e^{\beta X})e^{-\beta x}.$$

3. Let X be a Poisson random variable of parameter $\lambda \in (0, \infty)$.

(a) By optimizing the estimate of Question 2(b) over β, show that, for all $x \ge \lambda$,

$$\mathbb{P}(X \ge x) \le \exp\{-x\log(x/\lambda) - \lambda + x\}.$$

(b) Show, on the other hand, that, for integers x, as $x \to \infty$,

$$\mathbb{P}(X = x) \sim \frac{1}{\sqrt{2\pi x}}\exp\{-x\log(x/\lambda) - \lambda + x\}.$$

4. Consider a random sample $X_1, \dots, X_n$ taken from a distribution having mean μ and variance $\sigma^2 < \infty$. Use Chebyshev's inequality to determine a sample size n that will be sufficient, whatever the distribution, for the probability to be at least 0.99 that the sample mean $\bar{X}$ will be within two standard deviations of μ.

5. Suppose we conduct a sequence of independent Bernoulli trials and denote by X the number of trials up to and including the ath success. Show that

$$\mathbb{P}(X = r) = \binom{r-1}{a-1} p^a q^{r-a}, \quad r = a, a+1, \dots.$$

Show that the generating function for this distribution is $p^a t^a (1-qt)^{-a}$. Deduce that $\mathbb{E}(X) = a/p$ and $\mathrm{var}(X) = aq/p^2$. Explain how X can be represented as the sum of a independent random variables, all with the same distribution. Use this representation to derive again the mean and variance of X.

6. For a random variable X with mean μ and variance $\sigma^2 < \infty$, define the function

$$V(x) = \mathbb{E}((X-x)^2).$$

Express the random variable $V(X)$ in terms of μ, σ^2 and X, and hence show that

$$\mathbb{E}(V(X)) = 2\sigma^2.$$

7. Suppose X is a real-valued random variable and $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two non-decreasing functions. Prove the 'Chebyshev order inequality':

$$\mathbb{E}(f(X))\mathbb{E}(g(X)) \leq \mathbb{E}(f(X)g(X)).$$

[*Hint. Consider* $(f(X_1) - f(X_2))(g(X_1) - g(X_2))$ *where* X_1 *and* X_2 *are independent copies of* X.]

8. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent identically distributed random variables, with mean μ and variance $\sigma^2 < \infty$. Set $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Let N be a bounded non-negative integer-valued random variable which is independent of the sequence $(X_n : n \in \mathbb{N})$.

(a) Show that $\mathbb{E}(S_N) = \mu\mathbb{E}(N)$.

(b) Show that $\mathbb{E}(S_N^2 | N = n) = n\sigma^2 + n^2\mu^2$ and hence express the $\mathrm{var}(S_N)$ in terms of $\mathrm{var}(N)$.

Consider now the case where X_1 takes only the values 1 and -1. Fix a non-negative integer a and set

$$T = \min\{n \geq 0 : |S_n| = a\}.$$

(c) Show that $\mathbb{E}(S_T) = \mu\mathbb{E}(T)$ and find $\mathrm{var}(S_T)$.

9. At time 0, a blood culture starts with one red cell. At the end of one minute, the red cell dies and is replaced by one of the following combinations with probabilities as indicated:

$$\text{2 red cells } \tfrac{1}{4}, \quad \text{1 red, 1 white } \tfrac{2}{3}, \quad \text{2 white } \tfrac{1}{12}.$$

Each red cell lives for one minute and gives birth to offspring in the same way as the parent cell. Each white cell lives for one minute and dies without reproducing. Assume the individual cells behave independently.

(a) At time $n + \frac{1}{2}$ minutes after the culture began, what is the probability that no white cells have yet appeared?

(b) What is the probability that the entire culture dies out eventually?

10. Consider a population of animals in which each mature individual produces a random number of offspring with generating function F. Suppose we start with a population of k immature individuals, each of which grows to maturity with probability p, independently of the other individuals.

(a) Find the generating function for the distribution of the number of immature individuals in the next generation.

(b) Find the generating function for the distribution of the number of mature individuals in the next generation, given that there are k mature individuals in the parent generation.

(c) Show that the distributions in (a) and (b) have the same mean, but not necessarily the same variance.

11. A slot machine operates so that at the first turn the probability for the player to win is 1/2. Thereafter the probability for the player to win is 1/2 if he lost at the last turn, but is $p < 1/2$ if he won at the last turn. If u_n is the probability that the player wins at the nth turn, show that, provided $n > 1$,

$$u_n + (\tfrac{1}{2} - p)u_{n-1} = \tfrac{1}{2}.$$

Observe that this equation also holds for $n = 1$, if we set $u_0 = 0$. Solve the equation, showing that

$$u_n = \frac{1 + (-1)^{n-1}(\frac{1}{2} - p)^n}{3 - 2p}.$$

12. Let $F(t) = 1 - p(1-t)^\beta$, where $p \in (0,1)$ and $\beta \in (0,1)$ are constants. Show that $F(t)$ is the generating function of a probability distribution on $\mathbb{Z}^+$ and that its iterates are given by

$$F_n(t) = 1 - p^{1+\beta+\cdots+\beta^{n-1}}(1-t)^{\beta^n} \quad \text{for } n = 1, 2, \ldots.$$

Find the mean m of the associated distribution and the extinction probability of the branching process whose offspring distribution has generating function F.

13. Let $(X_n : n \in \mathbb{N})$ be a sequence of independent identically distributed random variables, with mean μ and variance $\sigma^2 < \infty$. Define

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^n X_i, \quad \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \hat{\mu}_n)^2.$$

Show that, for all $\varepsilon > 0$, as $n \to \infty$,

$$\mathbb{P}(|\hat{\mu}_n - \mu| > \varepsilon) \to 0$$

and, provided $\mathbb{E}(X_1^4) < \infty$, also

$$\mathbb{P}(|\hat{\sigma}_n^2 - \sigma^2| > \varepsilon) \to 0.$$

[*Hint. You may find it useful to show that*

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 - (\hat{\mu}_n - \mu)^2.]$$