Groups Example Sheet 1

Michaelmas 2016 Julia Goedecke
Please send comments and corrections to jg352.

The questions are not necessarily in order of difficulty. Particularly questions 12 and 13 are just questions on a later topic, so do make sure you look at them! You should attempt all questions on the sheet in any case and write up your best attempt/solution for each question to hand in. Starred or "exploration" questions should not be done to the detriment of later sheets or other subjects.

1. Let G be any group. Show that the identity e is the unique solution of the equation $a^2 = a$.

2. Let H_1 and H_2 be two subgroups of the group G.

   Show that the intersection $H_1 \cap H_2$ is a subgroup of G.

   Show that the union $H_1 \cup H_2$ is a subgroup of G if and only if one of the H_i contains the other.

3. Let G be a finite group.

   (a) Let $a \in G$. Show that there is a positive integer n such that $a^n = e$, the identity element. (The least such positive n is the *order* of a.)

   (b) Show that there exists a positive integer n such that $a^n = e$ for all $a \in G$. (The least such positive n is the *exponent* of G.)

4. Show that the set G of complex numbers of the form $\exp(i\pi t)$ with t rational is a group under multiplication (with identity 1). Show that G is infinite, but that every element a of G has finite order.

5. Let S be a finite non-empty set of non-zero complex numbers which is closed under multiplication. Show that S is a subset of the set $\{z \in \mathbb{C} : |z| = 1\}$. Show that S is a group, and deduce that for some $n \in \mathbb{N}$, S is the set of n-th roots of unity; that is, $S = \{\exp(2k\pi i/n) : k = 0, \ldots, n-1\}$.

6. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *)$ is a group. What is the inverse 2^{-1} of 2 in this group? Solve the equation $2 * x * 5 = 6$.

7. Let G be a group in which every element other than the identity has order two. Show that G is abelian. Show also that if G is finite, the order of G is a power of 2. [Consider a minimal generating set. A minimal generating set is a set which generates G but no proper subset of which generates G.]

8. Let G be a group of even order. Show that G contains an element of order two.

9. Let G be a finite group and f a homomorphism from G to H. Let $a \in G$. Show that the order of $f(a)$ is finite and divides the order of a.

10. Show that the dihedral group D_{12} is isomorphic to the direct product $D_6 \times C_2$. Is D_{16} isomorphic to $D_8 \times C_2$?

11. How many homomorphisms $D_{2n} \longrightarrow C_n$ are there? How many isomorphisms $C_n \longrightarrow C_n$?

12. Write these permutations as products of disjoint cycles and compute their order and sign:

    (a) (12)(1234)(12);

    (b) (123)(235)(345)(45).

13. What is the largest possible order of an element in S_5? And in S_9?
    Show that every element in S_{10} of order 14 is odd.

---

14. * Which groups contain a (non-zero) even number of elements of order 2?

15. * Let G be the set of integers modulo 2^n with operation

$$x * y = 4xy + x(-1)^y + y(-1)^x \pmod{2^n}$$

    Show that G is a cyclic group.

16. * *Exploration question* Is there any operation on the natural numbers $\mathbb{N}$ which makes it into a group? If yes, how many (non-isomorphic) such groups can you find?

    [If you want hints, you can look on the Moodle site in the forum.]