Groups Ia Practice Sheet B

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These questions are not supposed to form the work for one of the regular 4 groups supervisions, but instead they give you opportunities to practise getting used to axioms and definitions in your own time. If you find this useful, try to make similar questions for yourself on later material of the course.

Properties of general groups

In the following questions, let G be a group with operation $*$.

1. Without looking at your lecture notes, show that inverses are unique. That is, if $g, h \in G$ with $h * g = e = g * h$, show that $h = g^{-1}$.

2. Show that the equation $a * x = b$ has a unique solution for x in G, and find this solution.

3. For $a, b, c, d \in G$, use the associativity axiom to show that $((a * b) * c) * d = a * (b * (c * d))$. You will find similarly that all possible ways to bracket this product of four elements gives the same answer. Does this extend to products of five elements?

4. What is the inverse of $a * b * a^{-1}$? Simplify the expression $(a * b * a^{-1})^n$ as much as you can. [Here "to the power n" just means multiply (or "star") the expression in the brackets n times with itself.]

Subgroups

5. Show that the even numbers $2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\}$ form a subgroup of $\mathbb{Z}$. Show also that for any $n \in \mathbb{Z}$, the set $n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$ forms a subgroup of $\mathbb{Z}$.

6. (a) Show that the rotations of a regular triangle form a subgroup of all symmetries of the triangle. (You can also do this for a different or general n-gon if you wish.)

   (b) Show that the symmetries of a regular triangle form a subgroup of the symmetries of a regular hexagon. (How many of such subgroups are there in the symmetries of a regular hexagon?)

7. Show that $\{(x, 0, 0) \mid x \in \mathbb{R}\}$ is a subgroup of $\mathbb{R}^3$ (with addition). Show also that the sets $\{(x, x, x) \mid x \in \mathbb{R}\}$ and $\{(x, y, 0) \mid x, y \in \mathbb{R}\}$ are subgroups. Why is $\{(x, 1, 2) \mid x \in \mathbb{R}\}$ not a subgroup?