IA Groups: Example Sheet 3

1. Let G be a group. If H is a normal subgroup of G and K is a normal subgroup of H, is K a normal subgroup of G?

2. Let K be a normal subgroup of index m in the group G. Show that $g^m \in K$ for any element $g \in G$.

3. Let H be a subgroup of the cyclic group C_n. What is the quotient C_n/H?
Let D_{2n} be the group of symmetries of a regular n-gon. Show that any subgroup K of rotations is normal in D_{2n}, and identify the quotient D_{2n}/K.

4. Show that D_{2n} has two conjugacy classes of reflections if n is even, but only one if n is odd.

5. Let D_8 be the dihedral group of order 8. Find the conjugacy classes of D_8 and their sizes. Show that the centre Z of the group has order 2, and identify the quotient group D_8/Z of order 4.
Repeat with the quaternion group Q_8.

6. Let Q be a plane quadrilateral. Show that its group $G(Q)$ of symmetries has order at most 8. For which n in the set $\{1, 2, \dots, 8\}$ is there a quadrilateral Q with $G(Q)$ of order n?

7. Identify the elements of S_4 explicitly in its action as the group of rotations of a cube.

8. Show that the subgroup H of the group G is normal in G if and only if H is the union of some conjugacy classes of G.
Show that the symmetric group S_4 has a normal sugroup (usually denoted V_4) of order 4.
To which group of order 6 is the quotient group S_4/V_4 isomorphic?
Find an action of S_4 giving rise to this isomorphism.

9. Suppose that the group G acts on the set X. Let $x \in X$, let $y = g(x)$ for some $g \in G$. Show that the stabiliser G_y equals the conjugate gG_xg^{-1} of the stabiliser G_x.

10. Let G be a finite group and let X be the set of all subgroups of G. Show that G acts on X by $g : H \mapsto gHg^{-1}$ for $g \in G$ and $H \in X$, where $gHg^{-1} = \{ghg^{-1} : h \in H\}$. Show that the orbit containing H in this action of G has size at most $|G|/|H|$. If H is a proper subgroup of G, show that there exists an element of G which is contained in no conjugate gHg^{-1} of H in G.

11. Let G a finite group of prime power order p^a, with $a > 0$. By considering the conjugation action of G, show that the centre Z of G is non-trivial.
Show that any group of order p^2 is abelian, and that there are up to isomorphism just two groups of that order for each prime p.

12. Find the conjugacy classes of elements in the alternating group A_5, and determine their sizes.
Show that A_5 has no non-trivial normal subgroups (so A_5 is a *simple* group).
Show that if H is a proper subgroup of index n in A_5 then $n > 4$. [Consider the left coset action of A_5 on the set of left cosets of H in A_5.]

13. Let G be a finite group of order $p^a m$, where p^a is the highest power of the prome p dividing $|G|$. Let X be the set of all subsets of G of size p^a. Show that G acts on X by $g : A \mapsto gA$ for $g \in G$ and $A \in X$. Show that X has size prime to p, and deduce that there is a G-orbit X' of G on X of size prime to p. By considering the stabiliser of an element in X', show that G has a subgroup of order p^a, a *Sylow* subgroup of G.

Comments and corrections should be sent to `saxl@dpmms.cam.ac.uk`.