

ANALYSIS I EXAMPLES 2

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1. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 1 - x$ otherwise. Find $\{a : f \text{ is continuous at } a\}$.
 2. Write down the definition of “ $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ ”. Prove that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ if, and only if, $f(x_n) \rightarrow \infty$ for every sequence such that $x_n \rightarrow \infty$.
 3. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) \rightarrow \ell$ as $x \rightarrow a$ and $g(y) \rightarrow k$ as $y \rightarrow \ell$. Must it be true that $g(f(x)) \rightarrow k$ as $x \rightarrow a$?
 4. Let $f_n : [0, 1] \rightarrow [0, 1]$ be continuous, $n \in \mathbb{N}$. Let $h_n(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$. Show that h_n is continuous on $[0, 1]$ for each $n \in \mathbb{N}$. Must $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous?
 5. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$. Such a c is called a *fixed point* of f . Give an example of a bijection of $[0, 1]$ with no fixed point. If $h : (0, 1) \rightarrow (0, 1)$ is a continuous bijection, must it have a fixed point?
 6. Let $f(x) = \sin^2 x + \sin^2(x + \cos^7 x)$. Assuming the familiar features of \sin , \cos without justification, prove that there exists $k > 0$ such that $f(x) \geq k$ for all $x \in \mathbb{R}$.
 7. Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is continuous, that $f(0) = f(1) = 0$, and that for every $x \in (0, 1)$ there exists $0 < \delta < \min\{x, 1 - x\}$ with $f(x) = (f(x - \delta) + f(x + \delta))/2$. Show that $f(x) = 0$ for all x .
 8. Prove that $2x^5 + 3x^4 + 2x + 16 = 0$ has no real solutions outside $[-2, -1]$ and exactly one inside.
 9. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Which of (1)–(4) must be true?
 - (1) If f is increasing then $f'(x) \geq 0$ for all $x \in (a, b)$.
 - (2) If $f'(x) \geq 0$ for all $x \in (a, b)$ then f is increasing.
 - (3) If f is strictly increasing then $f'(x) > 0$ for all $x \in (a, b)$.
 - (4) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing.
- [*Increasing* means $f(x) \leq f(y)$ if $x < y$, and *strictly increasing* means $f(x) < f(y)$ if $x < y$.]
10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable for all x . Prove that if $f'(x) \rightarrow \ell$ as $x \rightarrow \infty$ then $f(x)/x \rightarrow \ell$. If $f(x)/x \rightarrow \ell$ as $x \rightarrow \infty$, must $f'(x)$ tend to a limit?

11. Let $f(x) = x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable everywhere and that $f'(0) = 1$, but that there is no interval around 0 on which f is increasing.

⁺**12.** Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Suppose that $f((x+y)/2) \leq (f(x) + f(y))/2$ for all $x, y \in [a, b]$. Prove that f is continuous on (a, b) . Must it be continuous at a and b too?

⁺**13.** Construct a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that takes every value on every interval – in other words for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.