

## ANALYSIS I EXAMPLES 4

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at [g.p.paternain@dpmmms.cam.ac.uk](mailto:g.p.paternain@dpmmms.cam.ac.uk).

1. Show directly from the definition of an integral that  $\int_0^a x^2 = a^3/3$  for  $a > 0$ .
2. Let  $f(x) = \sin(1/x)$  for  $x \neq 0$  and  $f(0) = 0$ . Does  $\int_0^1 f$  exist?
3. Give an example of a continuous function  $f : [0, \infty) \rightarrow [0, \infty)$ , such that  $\int_0^\infty f$  exists but  $f$  is unbounded.
4. Give an example of an integrable function  $f : [0, 1] \rightarrow \mathbb{R}$  with  $f \geq 0$ ,  $\int_0^1 f = 0$ , and  $f(x) > 0$  for some value of  $x$ . Show that this cannot happen if  $f$  is continuous.
5. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be monotonic. Show that  $\{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$  is countable. Let  $x_n, n \geq 1$  be a sequence of distinct points in  $(0, 1]$ . Let  $f_n(x) = 0$  if  $0 \leq x < x_n$  and  $f_n(x) = 1$  if  $x_n \leq x \leq 1$ . Let  $f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x)$ . Show that this series converges for every  $x \in [0, 1]$ . Show that  $f$  is increasing (and so is integrable). Show that  $f$  is discontinuous at every  $x_n$ .
6. Let  $f(x) = \log(1 - x^2)$ . Use the mean value theorem to show that  $|f(x)| \leq 8x^2/3$  for  $0 \leq x \leq 1/2$ . Now let  $I_n = \int_{n-1/2}^{n+1/2} \log x \, dx - \log n$  for  $n \in \mathbb{N}$ . Show that  $I_n = \int_0^{1/2} f(t/n) \, dt$  and hence that  $|I_n| \leq 1/9n^2$ . By considering  $\sum_{j=1}^n I_j$ , deduce that  $n!/n^{n+1/2}e^{-n} \rightarrow \ell$  for some constant  $\ell$ .  
[The bounds  $8x^2/3$  and  $1/9n^2$  are not best possible; they are merely good enough for the conclusion.]
7. Let  $I_n = \int_0^{\pi/2} \cos^n x$ . Prove that  $nI_n = (n-1)I_{n-2}$ , and hence  $\frac{2n}{2n+1} \leq I_{2n+1}/I_{2n} \leq 1$ . Deduce Wallis's Product:

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)} = \lim_{n \rightarrow \infty} \frac{2^{4n}}{2n+1} \binom{2n}{n}^{-2}.$$

By taking note of the previous exercise, prove that  $n!/n^{n+1/2}e^{-n} \rightarrow \sqrt{2\pi}$  (Stirling's formula).

8. Do these improper integrals converge? (i)  $\int_1^\infty \sin^2(1/x) \, dx$ , (ii)  $\int_0^\infty x^p \exp(-x^q) \, dx$  where  $p, q > 0$ .
9. Show that  $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \rightarrow \log 2$  as  $n \rightarrow \infty$ , and find  $\lim_{n \rightarrow \infty} \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{(-1)^{n-1}}{2n}$ .
10. Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous and suppose that  $\int_a^b f(x)g(x) \, dx = 0$  for every continuous function  $g : [a, b] \rightarrow \mathbb{R}$  with  $g(a) = g(b) = 0$ . Must  $f$  vanish identically?

**11.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Let  $G(x, t) = t(x - 1)$  for  $t \leq x$  and  $G(x, t) = x(t - 1)$  for  $t \geq x$ . Let  $g(x) = \int_0^1 f(t)G(x, t)dt$ . Show that  $g''(x)$  exists for  $x \in (0, 1)$  and equals  $f(x)$ .

**12.** Let  $I_n(\theta) = \int_{-1}^1 (1 - x^2)^n \cos(\theta x) dx$ . Prove that  $\theta^2 I_n = 2n(2n - 1)I_{n-1} - 4n(n - 1)I_{n-2}$  for  $n \geq 2$ , and hence that  $\theta^{2n+1}I_n(\theta) = n!(P_n(\theta) \sin \theta + Q_n(\theta) \cos \theta)$ , where  $P_n$  and  $Q_n$  are polynomials of degree at most  $2n$  with integer coefficients. Deduce that  $\pi$  is irrational.

**13.** Let  $f_1, f_2 : [-1, 1] \rightarrow \mathbb{R}$  be increasing and  $g = f_1 - f_2$ . Show that there exists  $K$  such that, for any dissection  $\mathcal{D} = x_0 < \cdots < x_n$  of  $[-1, 1]$ ,  $\sum_{j=1}^n |g(x_j) - g(x_{j-1})| \leq K$ . Now let  $g(x) = x \sin(1/x)$  for  $x \neq 0$  and  $g(0) = 0$ . Show that  $g$  is integrable but is not the difference of two increasing functions.

**14.** Show that if  $f : [0, 1] \rightarrow \mathbb{R}$  is integrable then  $f$  has infinitely many points of continuity.