ANALYSIS I EXAMPLES 2

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 1 - x$ otherwise. Find $\{a : f \text{ is continuous at } a\}$.

2. Write down the definition of "$f(x) \to \infty$ as $x \to \infty$". Prove that $f(x) \to \infty$ as $x \to \infty$ if, and only if, $f(x_n) \to \infty$ for every sequence such that $x_n \to \infty$.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $f(x) \to \ell$ as $x \to a$ and $g(y) \to k$ as $y \to \ell$. Must it be true that $g(f(x)) \to k$ as $x \to a$?

4. Let $f_n : [0, 1] \to [0, 1]$ be continuous, $n \in \mathbb{N}$. Let $h_n(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$. Show that h_n is continuous on $[0, 1]$ for each $n \in \mathbb{N}$. Must $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous?

5. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$. Such a c is called a *fixed point* of f. Give an example of a bijection of $[0, 1]$ with no fixed point. If $h : (0, 1) \to (0, 1)$ is a continuous bijection, must it have a fixed point?

6. Let $f(x) = \sin^2 x + \sin^2(x + \cos^7 x)$. Assuming the familiar features of sin without justification, prove that there exists $k > 0$ such that $f(x) \geq k$ for all $x \in \mathbb{R}$.

7. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, that $f(0) = f(1) = 0$, and that for every $x \in (0, 1)$ there exists $0 < \delta < \min\{x, 1 - x\}$ with $f(x) = (f(x - \delta) + f(x + \delta))/2$. Show that $f(x) = 0$ for all x.

8. Let $f : [a, b] \to \mathbb{R}$ be bounded. Suppose that $f((x + y)/2) \leq (f(x) + f(y))/2$ for all $x, y \in [a, b]$. Prove that f is continuous on (a, b). Must it be continuous at a and b too?

9. Prove that $2x^5 + 3x^4 + 2x + 16 = 0$ has no real solutions outside $[-2, -1]$ and exactly one inside.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Which of (1)–(4) must be true?

(1) If f is increasing then $f'(x) \geq 0$ for all $x \in (a, b)$.
(2) If $f'(x) \geq 0$ for all $x \in (a, b)$ then f is increasing.
(3) If f is strictly increasing then $f'(x) > 0$ for all $x \in (a, b)$.
(4) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing.

[*Increasing* means $f(x) \leq f(y)$ if $x < y$, and *strictly increasing* means $f(x) < f(y)$ if $x < y$.]

11. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable for all x. Prove that if $f'(x) \to \ell$ as $x \to \infty$ then $f(x)/x \to \ell$. If $f(x)/x \to \ell$ as $x \to \infty$, must $f'(x)$ tend to a limit?

12. Let $f(x) = x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable everywhere and that $f'(0) = 1$, but that there is no interval around 0 on which f is increasing.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also that for every rational r, the set S_r of all x with $f(x) = r$ is closed, that is, if x_n is any sequence in S_r with $x_n \to a$, then $a \in S_r$. Prove that f is continuous.