EXAMPLE SHEET 4

1. Show that the following Lie group actions are proper.
   (a) The action of $\mathbb{K}^*$ on $\mathbb{K}^{n+1} \setminus \{0\}$ by rescaling, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.
   (b) The right- (or left-) translation action of an embedded Lie subgroup $H \subset G$ on G.
   (c) Any action of a compact group.

2. (a) Given a vector bundle $\pi : E \to B$ and a collection of local $\mathfrak{gl}(k, \mathbb{R})$-valued 1-forms A_α satisfying the preliminary definition of a connection, show that the $\mathfrak{gl}(k, \mathbb{R})$-valued 1-form $\mathcal{A}$ constructed in lectures from the A_α is well-defined (i.e. consistent on overlaps) and satisfies the two conditions on a connection. *[Hint: Show that $D_b f_\beta = (R_{g_{\beta\alpha}^{-1}})_* D_b f_\alpha + f_\beta(b) \cdot \eta$, where $\eta = g_{\beta\alpha}\, \mathrm{d}g_{\beta\alpha}^{-1} \in \mathfrak{gl}(k, \mathbb{R})$.]*
   (b) Conversely show that if $\mathcal{A}$ is a connection then the $f_\alpha^* \mathcal{A}$ satisfy the preliminary definition.

3.† Let $\mathcal{A}$ be a connection on a vector bundle E.
   (a) Prove the Leibniz rule $\mathrm{d}^{\mathcal{A}}(fs) = f\, \mathrm{d}^{\mathcal{A}} s + s \otimes \mathrm{d}f$ for sections s and functions f.
   (b) Conversely, show that every $\mathbb{R}$-linear map $\mathcal{D} : \{\text{sections of } E\} \to \{E\text{-valued 1-forms}\}$ satisfying $\mathcal{D}(fs) = f\, \mathcal{D}s + s \otimes \mathrm{d}f$ is given by $\mathrm{d}^{\mathcal{A}}$ for a unique connection $\mathcal{A}$ on E.
   (c) Show that $(\mathrm{d}^{\mathcal{A}})^2 \sigma = F \wedge \sigma$ for any E-valued p-form σ.

4. Fix a G-bundle $\pi : P \to B$ with a connection $\mathcal{A}$.
   (a) Given vector fields v and w on B, let $\hat{\mathsf{v}}$ and $\hat{\mathsf{w}}$ denote their (unique) lifts to horizontal vector fields on P. Show that the vertical component of $[\hat{\mathsf{v}}, \hat{\mathsf{w}}]$ at a point p is $p \cdot -\mathcal{F}(\hat{\mathsf{v}}, \hat{\mathsf{w}})$.
   (b) Now take local coordinates on B around $\pi(p)$, and define $\gamma(t)$ to be the result of parallel transporting p for time t in the x^i-direction, then time t in the x^j-direction, then back round the other two sides of the square. Show that $\dot{\gamma}(0) = 0$ and $\ddot{\gamma}(0) = p \cdot -2\mathcal{F}(\mathsf{u}_i, \mathsf{u}_j)$, where u_i and u_j are any lifts of ∂_{x^i} and ∂_{x^j} to p. *[Hint: First do it for time u in the x^j-direction.]*

5. Recall the connection we defined on the Hopf bundle $S^{2n+1} \to \mathbb{CP}^n$ via its horizontal distribution $H_p = T_p S^{2n+1} \cap i \cdot T_p S^{2n+1}$. Trivialise the bundle over $U_0 \subset \mathbb{CP}^n$, and compute the local connection 1-form A and curvature F in this trivialisation.

6. Let $E \to B$ be a vector bundle of rank k, and G a Lie group equipped with a representation $\rho : G \to \mathrm{GL}(k, \mathbb{R})$. A *reduction of the structure group* of E to G comprises a G-bundle $P \to B$ and an isomorphism between E and the associated vector bundle $P \times_G \mathbb{R}^k$.
   (a) Show that a reduction of the structure group to $\mathrm{O}(k)$ is equivalent to a choice of inner product on E, via the orthogonal frame bundle $F_{\mathrm{O}}(E)$.
   (b) Show that a connection $\mathcal{A}$ on E is compatible with a given inner product iff it's induced from a connection on $F_{\mathrm{O}}(E)$.

7.† Let (X, g) be a Riemannian manifold, equipped with an arbitrary connection whose local connection 1-forms have components $\Gamma^i{}_{jk}$.
   (a) Find coordinate expressions for ∇g and the torsion T, and deduce that the Christoffel symbols are given by $\Gamma_{kij} = \frac{1}{2}(\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij})$.

   Now assume that the connection is the Levi-Civita connection.
   (b) Given a vector field v on X, show that in coordinates we have
   $$(\mathcal{L}_{\mathsf{v}} g)_{ij} = \partial_i(g_{kj} v^k) + \partial_j(g_{ik} v^k) - v^k \Gamma_{kij}.$$
   (c) For points p and q in X, let $\mathcal{P}$ be the space of smooth paths $[0, 1] \to X$ from p to q, and define the energy functional $E : \mathcal{P} \to \mathbb{R}$ by
   $$E(\gamma) = \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))\, \mathrm{d}t.$$
   Suppose $\gamma \in \mathcal{P}$ is a stationary point of E. By considering perturbations of γ given by flowing along a vector field vanishing at p and q, show that γ must satisfy the *geodesic equation*
   $$\ddot{\gamma}^k + \Gamma^k{}_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0.$$

8. (a) Show that for vector fields v and w on a manifold X, equipped with a connection, we have

$$\nabla_{\mathsf{v}}\mathsf{w} - \nabla_{\mathsf{w}}\mathsf{v} = [\mathsf{v}, \mathsf{w}] + T(\mathsf{v}, \mathsf{w}),$$

where T is the torsion of the connection.

(b) Show that the Riemann tensor of a Riemannian manifold (X, g) vanishes iff X can be covered by coordinate patches on which $g = \sum_i (\mathrm{d}x^i)^2$. Such a metric is called *flat*. *[Hint: Use the fact (proved in the third Examples Class) that a fibrewise basis of vector field* v_i *arises as coordinate vector fields* ∂_{x^i} *iff* $[\mathsf{v}_i, \mathsf{v}_j] = 0$ *for all* i *and* j*.]*

9. Let (X, g) be a compact oriented Riemannian n-manifold.

(a) Show that a p-form α is harmonic if and only if it is closed and coclosed. *[Hint: for one direction consider* $\langle \alpha, \Delta\alpha \rangle_X$*.]*

(b) By considering harmonic representatives, construct an isomorphism $H^p_{\mathrm{dR}}(X) \to H^{n-p}_{\mathrm{dR}}(X)$ for each p.

10.* Let (X, g_X) be a Riemannian manifold and $\iota : Y \to X$ an embedded submanifold equipped with the metric $g_Y = \iota^* g_X$. Let $\mathcal{A}_X$ be the Levi-Civita connection on X, and let $\mathcal{A}_Y$ be the connection on Y induced from $\mathcal{A}_X$ by the splitting $\iota^* TX = TY \oplus TY^\perp$. Show that $\mathcal{A}_Y$ is torsion-free and compatible with g_Y, and hence is the Levi-Civita connection on Y.

11.* Consider $\mathbb{R}^3$ with its standard metric and orientation. Express div, grad, and curl in terms of: the exterior derivative, the Hodge star operator, and raising and lowering indices.

j.smith@dpmms.cam.ac.uk