EXAMPLE SHEET 2

1. Let T, U, and V be finite-dimensional vector spaces over a field $\mathbb{K}$. Let $\mathsf{e}_1, \dots, \mathsf{e}_m$ be a basis for U, with dual basis $\varepsilon_1, \dots, \varepsilon_m$, and let $\mathsf{f}_1, \dots, \mathsf{f}_n$ be a basis for V.
   (a) Show that there is a canonical isomorphism $V \otimes U^\vee \cong \mathcal{L}(U, V)$.
   (b) Given $\alpha \in \mathcal{L}(U, V)$, viewed as an element of $V \otimes U^\vee$, what is the meaning of its components with respect to the basis $\mathsf{f}_j \otimes \varepsilon_i$?
   (c) Show that contraction $V \otimes U^\vee \otimes U \otimes T^\vee \to V \otimes T^\vee$ corresponds to the composition map $\mathcal{L}(U, V) \otimes \mathcal{L}(T, U) \to \mathcal{L}(T, V)$.

2. (a) Recall the Möbius bundle $M \to S^1$ from Sheet 1. Show that $M \oplus M$ is trivial.
   (b) Show that for all n we have $TS^n \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}$ over S^n. *[Hint: View* $\underline{\mathbb{R}}^{n+1}$ *as* $\iota^* T\mathbb{R}^{n+1}$*, where* $\iota : S^n \to \mathbb{R}^{n+1}$ *is the inclusion.]*

3. Show that $H^*\mathcal{O}_{\mathbb{CP}^n}(-1) \cong \underline{\mathbb{C}}$, where $H : S^{2n+1} \to \mathbb{CP}^n$ is the Hopf map.

4.† Let α be a nowhere-zero 1-form on a manifold X, and let β be a p-form.
   (a) Show that if $\beta = \alpha \wedge \gamma$ for some $(p-1)$-form γ then $\alpha \wedge \beta = 0$.
   (b) Conversely, show that if $\alpha \wedge \beta = 0$ then there exists a $(p-1)$-form γ with $\beta = \alpha \wedge \gamma$. *[Hint: work locally in a coordinate patch, then use a partition of unity.]*
   (c) Must we have $\beta \wedge \beta = 0$?

5.† (a) Show that $H^1_{\mathrm{dR}}(S^n) = 0$ for $n \geq 2$ by taking a closed 1-form α and considering its restrictions to $U_\pm = S^n \setminus \{(0, \dots, 0, \pm 1)\}$.
   (b) For $n \geq 1$ use a similar idea to prove by induction that

$$H^r_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R} & \text{if } r = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

   Recall that we did $n = 1$ in lectures. *[Hint:* $U_+ \cap U_-$ *is homotopy equivalent to* S^{n-1}*.]*

6. Show that the tautological bundle $\mathcal{O}_{\mathbb{RP}^1}(-1)$ is non-orientable. By considering a suitable map $F : \mathbb{RP}^1 \to \mathbb{RP}^n$, show that $\mathcal{O}_{\mathbb{RP}^n}(-1)$ is also non-orientable.

7. Let X be a connected orientable n-manifold.
   (a) Show that X has exactly two orientations, and hence define what it means for a diffeomorphism $F : X \to X$ to be orientation-preserving or orientation-reversing.
   (b) For a compactly-supported n-form ω on X, show that

$$\int_X F^*\omega = \pm \int_X \omega,$$

   where the $\pm$ is the *orientation sign* of F ($+$ if orientation-preserving, $-$ if orientation-reversing).

8. (a) For which n is the antipodal map $\alpha : S^n \to S^n$ orientation-preserving?
   (b) By considering a natural $2:1$ map $\pi : S^n \to \mathbb{RP}^n$, deduce the values of n for which $\mathbb{RP}^n$ is orientable. You may assume without proof that π is locally a diffeomorphism.
   (c) By combining these ideas with Q5. and Q7. compute $H^n_{\mathrm{dR}}(\mathbb{RP}^n)$ for all n.

9. Let Σ be a compact 2-manifold-with-boundary, let $F : \Sigma \to \mathbb{R}^2$ be a smooth map, and let α be the 1-form $P\mathrm{d}x + Q\mathrm{d}y$ on $\mathbb{R}^2$ (where P and Q are smooth functions). Prove a version of Green's theorem by applying Stokes to $F^*\alpha$.

10.* The *hairy ball theorem* states that for even $n \geq 2$ there is no nowhere-zero vector field on S^n (this implies that TS^n is non-trivial). Fix such an n and prove the theorem as follows.
   (a) Show directly that a nowhere-zero vector field on S^n induces a homotopy from the identity to the antipodal map.
   (b) Use de Rham cohomology to prove that no such homotopy exists.

j.smith@dpmms.cam.ac.uk