EXAMPLE SHEET 1

1. Let X be a topological n-manifold and let $\mathbb{A}$ and $\mathbb{B}$ be smooth atlases on X.
   (a) Show that if $\mathbb{A}$ and $\mathbb{B}$ are smoothly equivalent then they have the same smooth functions, in the following sense: for any open $U \subset X$, a function $f : U \to \mathbb{R}$ is smooth with respect to $\mathbb{A}$ if and only if it's smooth with respect to $\mathbb{B}$.
   (b) Prove the converse. *[Hint: Consider local coordinate functions.]*

2. Show that the homogeneous coordinate maps on $\mathbb{RP}^n$ and $\mathbb{CP}^n$ define smooth pseudo-atlases. Show that resulting spaces are Hausdorff and second-countable, and hence are smooth manifolds.

3. Suppose X and Y are smooth manifolds with respective atlases
$$\{\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha\}_{\alpha\in\mathcal{A}} \quad \text{and} \quad \{\psi_\beta : S_\beta \to T_\beta\}_{\beta\in\mathcal{B}}.$$
Show that a map $F : X \to Y$ is smooth if and only if there exists a cover $\{W_\gamma\}_{\gamma\in\mathcal{C}}$ of X, and for each $\gamma \in \mathcal{C}$ there exist elements $\alpha(\gamma) \in \mathcal{A}$ and $\beta(\gamma) \in \mathcal{B}$, such that for all γ we have:
   (a) W_γ is contained in $U_{\alpha(\gamma)}$ and $F(W_\gamma)$ is contained in $S_{\beta(\gamma)}$.
   (b) $\varphi_{\alpha(\gamma)}(W_\gamma)$ is open in $V_{\alpha(\gamma)}$. (This is equivalent to W_γ being open in X, but we have phrased it this way so as not to mention the topology on X explicitly.)
   (c) The map
$$\psi_{\beta(\gamma)} \circ F \circ \varphi_{\alpha(\gamma)}|^{-1}_{W_\gamma} : \varphi_{\alpha(\gamma)}(W_\gamma) \to T_{\beta(\gamma)}$$
   is smooth.

4. Using Q3. show that the Hopf map $H : S^{2n+1} \to \mathbb{CP}^n$ is smooth.

5. About which points in $\mathbb{R}^2$ do the functions x and $r = \sqrt{x^2+y^2}$ provide local coordinates? (I.e. about which points can we use (x, r) as a chart?) Draw the corresponding basis vectors ∂_x and ∂_r at a representative selection of points.

6. Write down a smooth homeomorphism $\mathbb{R} \to \mathbb{R}$ that is not a diffeomorphism.

7.† Let $\iota : S^2 \to \mathbb{R}^3$ be the inclusion, and let $F : \mathbb{CP}^1 \to S^2$ be the map defined by
$$[z_0 : z_1] \mapsto \frac{1}{\|\mathbf{z}\|^2}\left(2\overline{z}_0 z_1, |z_1|^2 - |z_0|^2\right) \in S^2 \subset \mathbb{C} \oplus \mathbb{R} = \mathbb{R}^3.$$
Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$ and let (u, v) be coordinates parametrising $U_0 = \{z_0 \neq 0\} \subset \mathbb{CP}^1$ via $(u, v) \mapsto [1 : u + iv]$.
   (a) Compute the derivative of $\iota \circ F$ on U_0 in terms of these coordinates.
   (b) Show that F is a diffeomorphism, so $\mathbb{CP}^1$ is a sphere (the Riemann sphere).

8.† Define the *Möbius bundle* $M \to S^1$ to be the line bundle trivialised over $U_\pm = S^1 \setminus \{(0, \pm 1)\}$ with transition function
$$g_{+-} : S^1 \setminus \{(0, \pm 1)\} \to \mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^*$$
given by 1 on the left-hand semicircle and -1 on the right-hand semicircle.
   (a) Show that $\mathbb{RP}^1$ is diffeomorphic to S^1.
   (b) Using this diffeomorphism to identify $\mathbb{RP}^1$ with S^1, show that M is isomorphic to the tautological bundle $\mathcal{O}_{\mathbb{RP}^1}(-1)$.

9. Using the two standard stereographic projection charts, trivialise TS^2 over $U_\pm = S^2 \setminus \{0, 0, \pm 1\}$, and calculate the transition function $g_{+-} : U_+ \cap U_- \to \mathrm{GL}(2, \mathbb{R})$. Deduce that TS^2 is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(2)$ as rank-2 real vector bundles (i.e. viewing the fibres of $\mathcal{O}_{\mathbb{CP}^1}(2)$ as $\mathbb{R}^2$ instead of $\mathbb{C}$).

10.* Fix a manifold X and a point p in X. Let $\mathcal{O}_{X,p}$ be the space of germs of smooth functions about p; this is an $\mathbb{R}$-vector space by scalar multiplication. An $\mathbb{R}$-*linear derivation* $\mathcal{O}_{X,p} \to \mathbb{R}$ is an $\mathbb{R}$-linear map $d : \mathcal{O}_{X,p} \to \mathbb{R}$ such that $d(fg) = d(f)g(p) + f(p)d(g)$ for all f and g. These form an $\mathbb{R}$-vector space denoted by $\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{X,p}, \mathbb{R})$. Show that this vector space is naturally isomorphic to T_pX.

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