Representation Theory — Examples Sheet 1

1. Let ρ be a representation of a group G. Show that $\det\rho$ is a representation of G. What is its degree?

2. Let θ be a one-dimensional representation of a group G and $\rho\colon G \to GL(V)$ another representation of G. Show that the map $\theta\otimes\rho\colon G\to GL(V)$ given by $\theta\otimes\rho(g) = \theta(g)\cdot\rho(g)$ defines a representation of G. If ρ is irreducible, must $\theta\otimes\rho$ also be irreducible?

3. Suppose that N is a normal subgroup of a group G. Given a representation of the quotient group G/N on a vector space V, explain how to construct an associated representation of G on V. Which representations of G arise in the way? Recall that G' is the normal subgroup of G generated by all elements of the form $ghg^{-1}h^{-1}$ with $g,h\in G$. Show that the 1-dimensional representations of G are precisely those that arise from 1-dimensional representations of G/G'.

4. Suppose that (ρ,V) and (σ,W) are representations of a group G. Show that $(\tau,\mathrm{Hom}_k(V,W))$ is a representation of G where $\tau(g)(\alpha) := \sigma(g)\circ\alpha\circ\rho(g^{-1})$ for all $g\in G$ and $\alpha\in\mathrm{Hom}_k(V,W)$. Show that $\mathrm{Hom}_G(V,W)$ is a subrepresentation, on which the elements of G act trivially.

5. Let C_n be the cyclic group of order n. Explicitly decompose the complex regular representation $\mathbb{C}C_n$ as a direct sum of irreducible subrepresentations.

6. Let $\rho\colon\mathbb{Z}\to GL_2(\mathbb{C})$ be the representation defined by $\rho(1) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. Show that ρ is not completely reducible. By a similar construction, show that if k is a field of characteristic p there is a two dimensional k-representation of C_p that is not completely reducible. *Optional: If k algebraically closed and of characteristic p, for which n is every k-representation of the cyclic group C_n completely reducible?*

7. Let D_{10} be the dihedral group of order 10. This is generated by elements r,t, with $r^5 = 1 = t^2$, $trt = r^{-1}$.. Let V be an irreducible complex representation of D_{10}. Pick an eigenvector v for r, and show the subspace spanned by v, tv is preserved by the action of D_{10}. Hence conclude that every irreducible complex representation of D_{10} has degree 1 or 2. By describing them explicitly, show that there are precisely four such representations up to isomorphism. Show moreover that for each such representation it is possible to choose a basis so that all the representing matrices have real entries.

8. Show that (up to isomorphism) there is only one irreducible complex representation of Q_8 of dimension at least two. Show that this representation cannot be realised over $\mathbb{R}$ and deduce that that Q_8 is not isomorphic to a subgroup of $GL_2(\mathbb{R})$. Find a four-dimensional irreducible real representation V of Q_8.

9. We showed in class that Jordan normal form implies that a representation of a finite *cyclic* group on a complex vector space V is a direct sum of one dimensional representations. Using this, show that this is still true for finite *abelian* groups G. [Hint: Pick $g\in G$. Show that the decomposition of V into eigenspaces of g is preserved by all $h\in G$. Now pick $h\in G\setminus\{g\}$, and decompose each eigenspace of g into eigenspaces of h. Repeat.] In particular, all irreducible representations are one dimensional.

10. Using the description of an abelian group as a product of cyclic groups, describe each one dimensional complex representation of a finite abelian group G. Show that there are exactly $|G|$ such. *Optional: When does such a G have a faithful irreducible representation?*

11. Let G act on a complex representation V, and let $v\in V$. Show that averaging over the orbit of G on v defines a vector $\overline{v}$ fixed by G. Now suppose V is an irreducible representation of G, and V is not the trivial representation. What can you say about $\overline{v}$?

12. *Optional question.* Let G act on a complex representation V, and let $\langle,\rangle : V\times V\to\mathbf{C}$ be a skew-symmetric (= alternating) form, that is $\langle x,y\rangle = -\langle y,x\rangle$ for all $x,y\in V$. Show that $(x,y) := 1/|G|\sum_{g\in G}\langle gx,gy\rangle$ is a G-invariant skew-symmetric form.

    Does this imply every finite subgroup of $GL_{2n}(\mathbf{C})$ is conjugate to a subgroup of $Sp_{2n}(\mathbf{C}) = \{g\in GL_{2n} \mid \langle gx,gy\rangle_{std} = \langle x,y\rangle_{std} \text{ for all } x,y\}$? Here $\langle e_i, e_{2n+1-j}\rangle_{std} = \delta_{ij}$, $\langle e_i,e_j\rangle_{std} = \langle e_{i+n},e_{j+n}\rangle_{std} = 0$, $1\le i,j\le n$ is the 'standard' skew-symmetric form.

13. Show that if $\rho\colon G\to GL(V)$ is a representation of a finite group G on a real vector space V then there is a basis for V with repect to which the matrix representing $\rho(g)$ is orthogonal for every $g\in G$. Which finite groups have a faithful two-dimensional real representation?