KUWAIT FOUNDATION LECTURE 56 - May 17, 2006

The Tamagawa Number Conjecture

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The talk gave a gentle introduction to the conjectural picture on leading Taylor coefficients of motivic L-functions, nowadays subsumed under the heading of the (equivariant) Tamagawa number conjecture, which is the culmination of a long historical development beginning with formulae of Leibeniz, Euler, Gauss, Dirichlet, Eisenstein, continuing with Dedekind's general class number formula for number fields, the conjecture of Birch and Swinnerton-Dyer (BSD), Borel's theorem on the Dedekind's ζ-function at positive integers, conjectures of Deligne, Bloch, Beilinson, Bloch-Kato, Fontaine and Perrin-Rion, Burns and myself and others. Much current work in number theory entirely concerns itself with the conjecture of BSD for elliptic curves over $\mathbb{Q}$ and their base change to number fields as extremely powerful techniques are available in this case after work of Gross-Zagier, Kolyvagin, Kato, Taylor-Wiles et al, Darmon and Zhang. Such curves, however, only constitute a small class among all arithmetic schemes and, after reviewing some history, the emphasis of the talk was to present what is expected in general. If $L(M,s)$ the Taylor expansion $L(M,s) = L^*(M)S^{r(M)} + .-$ at $s = 0, r(M)\epsilon\mathbb{Z}$ is described as an Euler characteristic of motivic cohomology and the leading coefficient $L^*(M)\epsilon\mathbb{R}$ is described in terms of the 'fundamental line' $\equiv (M)$, a determinant line bundle of various rational cohomology spaces associated to M. There are isomorphisms $\vartheta_\infty :\equiv (M) \otimes \mathbb{R} \simeq \mathbb{R}$ and $\vartheta_l :\equiv (M) \otimes \mathbb{Q}_l \simeq det_{\mathbb{Q}l} R\lceil_C(\mathbb{Z}C[\frac{1}{S}],\ M_l)$ for each prime l and one conjectures $\vartheta_\infty^{-1}(L^*(M))E \equiv (M) \otimes 1$ and $\vartheta_l \vartheta_\infty^{-1}(L(M))$ generates the $\mathbb{Z}_l$-lattice $\det_{\mathbb{Z}_l} R\lceil_C(\mathbb{Z}C[\frac{1}{S}]\zeta, T_l)$ where $T_l\ CM_l$ is any $G_\mathbb{Q}$ stable $\mathbb{Z}_l$-lattice. It is quite suggestive to expect that $\equiv (M)$ is the determinant of a naturally occurring complex and some recent ideas of Lichtenbaum ('Weil-étale cohomology') in this direction were presented. Further evidence for this point of view is the extension of this conjecture to an equivalent context where a finite-dimensional semisimple $\mathbb{Q}$-algebra A acts on M. The simplest example to look at is the motive of a point where the conjecture for $A = \mathbb{Q}$ is equivalent to the analytic class number formula and hence known but where the conjecture for $A = \mathbb{Q}[G], G = Gal(F/\mathbb{Q})$ is to a large extent open. By explicit classical formulae for Dirichlet L-functions one can treat the case where G is abelian (the subtlety coming from the fact that one needs to describe the $\mathbb{Z}_l[G]$-lattice spawned by the L-value). For l odd this was done by Burns and Geither. The case $l = 2$ was discussed at the end of the talk.

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The talk gave a gentle introduction to the conjectural picture on leading Taylor coefficients of motivic L-functions, nowadays subsumed under the heading of the (equivariant) Tamagawa number conjecture, which is the culmination of a long historical development beginning with formulae of Leibniz, Euler, Gauß, Dirichlet, Eisenstein, continuing with Dedekind's general class number formula for number fields, the conjecture of Birch and Swinnerton-Dyer (BSD), Borel's theorem on the Dedekind ζ-function at positive integers, conjectures of Deligne, Bloch, Beilinson, Bloch-Kato, Fontaine and Perrin-Riou, Burns and myself and others. Much current work in number theory entirely concerns itself with the conjecture of BSD for elliptic curves over $\mathbb{Q}$ and their base change to number fields as extremely powerful techniques are available in this case after work of Gross-Zagier, Kolyvagin, Kato, Taylor-Wiles et al, Darmon and Zhang. Such curves, however, only constitute a small class among all arithmetic schemes, and after reviewing some history, the emphasis of the talk was to present what is expected in general. If $L(M,s)$ has the Taylor expansion $L(M,s) = L^*(M)s^{r(M)} + \dots$ at $s=0$, $r(M)\in\mathbb{Z}$ is described as an Euler characteristic of motivic cohomology and the leading coefficient $L^*(M)\in\mathbb{R}$ is described in terms of the "fundamental line" $\Xi(M)$, a determinant line bundle of various rational cohomology spaces associated to M. There are isomorphisms $\vartheta_\infty: \Xi(M)\otimes\mathbb{R}\simeq\mathbb{R}$ and $\vartheta_\ell: \Xi(M)\otimes\mathbb{Q}_\ell \simeq \det_{\mathbb{Q}_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], M_\ell)$ for each prime ℓ and one conjectures $\vartheta_\infty^{-1}(L^*(M))\in\Xi(M)\otimes 1$ and $\vartheta_\ell\vartheta_\infty^{-1}(L^*(M))$ generates the $\mathbb{Z}_\ell$-lattice $\det_{\mathbb{Z}_\ell} R\Gamma_c(\mathbb{Z}[\frac{1}{S}], T_\ell)$ where $T_\ell\subset M_\ell$ is any $G_{\mathbb{Q}}$-stable $\mathbb{Z}_\ell$-lattice. It is quite suggestive to expect that $\Xi(M)$ is the determinant of a naturally occurring complex and some recent ideas of Lichtenbaum ("Weil-étale cohomology") in this direction were presented. Further evidence for this point of view is the extension of this conjecture to an equivariant context where a finite-dimensional semisimple $\mathbb{Q}$-algebra A acts on M. The simplest example to look at is the motive of a point where the conjecture for $A=\mathbb{Q}$ is equivalent to the analytic class number formula and hence known but where the conjecture for $A=\mathbb{Q}[G]$, $G=\mathrm{Gal}(F/\mathbb{Q})$ is to a large extent open. By explicit classical formulae for Dirichlet L-functions one can treat the case where G is abelian (the subtlety coming from the fact that one needs to describe the $\mathbb{Z}_\ell[G]$-lattice spanned by the L-value). For ℓ odd this was done by Burns and Greither. The case $\ell=2$ was discussed at the end of the talk. MF