What is curvature? W.T.G.

1. Introduction.

In this brief handout I shall follow the advice of the schedules and discuss curvature *informally*. This means that I will not get as far as a complete definition of curvature that can be applied to an arbitrary Riemannian manifold, or even an arbitrary surface with a Riemannian metric. However, I hope to convey how this might be done. Dr Shepherd-Barron's notes from two years ago (available on the web) go rather further, but also rather more quickly.

What we are trying to do is make precise the idea that not every surface can be flattened, and that there are two potential reasons for this. If you try to flatten the surface of a sphere, then you find that you haven't got enough stuff at the edges, and you end up tearing it. On the other hand, if you take a saddle-shaped surface (the soggy Pringle I mentioned in lectures) then you have too much, and find that you have to fold bits over other bits.

Notice, before we get on to this, that the surface of a cylinder is *flat*, according to this rough principle. Of course, any non-mathematician would call it curved, but from the mathematical point of view it is flat because it can be flattened without distortion. (To be more accurate, any small piece of it can, or indeed any simply connected piece - that is any piece not containing a closed path that goes right round the cylinder.) This is reflected in the everyday fact that it *really is* convenient to wrap wallpaper round cardboard cylinders.

Another way to argue for the flatness of a cylinder is to draw attention to the fact that when we discuss its geometry we are thinking about the *geodesic metric* (distances measured by shortest path *within the surface*) and not the distance in the space $\mathbb{R}^3$ in which the cylinder happens to live. It is pretty easy to see that the geodesic metric feels absolutely no evidence of any curviness.

2. Curves in Space.

I hope this section will be revision from Vector Calculus. In any case, I shall be brief. Let $\phi : [a, b] \to \mathbb{R}^3$ be a curve in space and make the following extra assumptions. First, ϕ is *sufficiently smooth*, by which I mean that I shall differentiate it without worrying, and second, it is *parametrized by arc length*, by which I mean that $\phi(t)$ moves at a constant speed of one. To be more precise, if $\phi(t) = (u(t), v(t), w(t))$, then $\phi'(t) = (u'(t), v'(t), w'(t))$ is a unit vector for all t. It follows from this that $\phi''(t) = (u''(t), v''(t), w''(t))$ is orthogonal to $\phi'(t)$ for every t. (This is obvious geometrically - if $\phi'(t)$ is constrained to lie on the surface of the unit sphere, then $\phi''(t)$, its rate of change, must be tangent to that sphere, and hence orthogonal to $\phi'(t)$.

Now we would like to define the *radius of curvature of* ϕ at t. Basically, what this means is that we fit the best circle we can to ϕ, touching it at the point $\phi(t)$, and work out its radius. This is done by considering second derivatives. Just as the first derivative gives you the best linear approximation to a curve, the second derivative gives you the best quadratic one, via the first three terms of the Taylor expansion. Matching that up with the best quadratic approximation of a circle gives you the circle.

To see how this leads to an easy calculation, consider a circle of radius R. To parametrize this by arc length we take $\phi(t) = (R\cos(t/R), R\sin(t/R))$. Then $\phi'(t) = (-\sin(t/R), \cos(t/R))$, which is a unit vector, as we wanted, and $\phi''(t) = R^{-1}(-\cos(t/R), -\sin(t/R))$, which has size $1/R$. Now the best second-order approximation to *any* curve ϕ (near t say) is

$$\phi(t+u) = \phi(t) + u\phi'(t) + \frac{1}{2}u^2\phi''(t) + o(u^3) \ .$$

It follows that a sensible definition for the radius of curvature at a point t is $1/\|\phi''(t)\|$.

3. Surfaces in Space.

For this section I shall be very sketchy indeed. Given a surface S in $\mathbb{R}^3$ and a point $x \in S$ we would like to attach a number to x, called *the curvature of* S *at* x. This is done as follows. Given any geodesic through x, you can work out its radius of curvature, and also attach a direction, which will be normal to the surface (a fact that needs to be proved). It will always be possible to find two orthogonal *principal directions* through x (roughly, orthogonal eigenvectors to a certain symmetric 2×2 matrix of second partial derivatives). In the case of a point on a sphere, these could be any two orthogonal directions. At a saddle point, one would curve upwards as steeply as possible, and the other downwards as steeply as possible. On the surface of a vertical cylinder, one would be horizontal and the other vertical. Taking geodesics through these and working out their radii of curvature at x, one gets a non-negative number. If the corresponding directions are the same (as e.g. in the sphere) then leave that number alone. If they are different (as e.g. at a saddle point) then take minus that number. (In other words, take the scalar product of the two vectors from x to the "centres" of the curves at x.) This gives a real number, and the reciprocal of that number is defined as the curvature at x. This is $1/R^2$ for a sphere of radius R and is *negative* at a saddle point. Notice that a bigger radius means a smaller curvature, which if you think about it makes good sense.

4. An Intrinsic Definition.

The above discussion is *extrinsic* in the sense that it depended very much on the embedding of the surface S into $\mathbb{R}^3$. It turns out that the *answer* does not depend on the embedding (a famous result of Gauss). This suggests that there ought to be an *intrinsic* definition - one that depends only on the geometry of the surface itself, by which I mean one that could be observed by some two-dimensional creature that was confined to the surface.

Such a definition is desirable for aesthetic reasons, but also for mathematical ones, as nothing I have said so far gives any clue about how to work out the curvature of the hyperbolic plane, which was discussed without reference to a particular embedding. (Even the hyperboloid model wasn't, as it did not inherit its metric from the Euclidean metric on $\mathbb{R}^3$.)

What sort of property, observable from within a surface, could serve as a definition of curvature? All we have to do to answer that question is look at results of the course. In a positively curved surface, angles of triangles add to more than π, whereas in a negatively curved one (assuming for a moment that the hyperbolic plane *is* negatively curved) they add to less. In a positively curved surface, lines that start out in the same direction get closer and closer. In a negatively curved surface they start to diverge.

I shall give a calculation for the hyperbolic plane based on the fact that small circles have a circumference that is larger than it would be in the Euclidean case.

First, let us consider the sphere. A circle of spherical radius r has circumference $2\pi \sin r$ (since the circumference equals its Euclidean circumference) which, for small r, is approximately $2\pi(r - r^3/6)$. Now suppose we had instead started with a sphere of radius R. A circle of radius r in this surface has Euclidean radius $R\sin(r/R)$ (as is easy to check) which is approximately $2\pi(r - r^3/6R^2)$. Remembering that the curvature of such a sphere is R^2, an obvious definition of the curvature of an arbitrary surface is this: measure the circumference of a circle of small radius r. If it is $2\pi(r - \kappa r^3/6)$, then the curvature is κ.

Now let us try to do the calculation in the hyperbolic plane. We shall work with the disc model, and consider a circle of (hyperbolic) radius r about 0. This has Euclidean radius $\tanh(r/2)$ (since the hyperbolic distance from 0 to ρ is $2\tanh^{-1}\rho$) and its circumference is therefore the length of the path $\phi(t) = \exp(it)\tanh(r/2)$ (t from 0 to 2π). This length is

$$\int_0^{2\pi} \frac{2|\phi'(t)|dt}{1-\tanh^2(r/2)} = \frac{4\pi\tanh(r/2)}{1-\tanh^2(r/2)} = \frac{4\pi\tanh(r/2)}{\operatorname{sech}^2(r/2)} = 2\pi\sinh r\ .$$

(Alternative "proof" - if it's $2\pi\sin r$ in the spherical case, then what else is it going to be in the hyperbolic case?) Since $\sinh r$ is approximately $r + r^3/6$ when r is small, we find that the curvature of the hyperbolic plane at 0 is -1. By symmetry, the hyperbolic plane has curvature -1 everywhere.

Just as a final remark, it can be shown that for a *conformal* Riemannian metric - that is, one of the form $\rho(z)|dz|$, a sensible definition of curvature is $\frac{-\nabla^2\log\rho(z)}{\rho(z)^2}$. A more sophisticated definition for general Riemannian metrics also exists.