Inverse Problems in Geometry and Dynamics
Lecture notes

Will J. Merry and Gabriel P. Paternain

March 2011

This is a complete, but not yet fully proofread, version. We would greatly appreciate any help in pointing out our mistakes.

Contents

Preface

These notes are based on lectures given by the second author for Part III of the Mathematical Tripos at the University of Cambridge. The aim is to expose the students to a circle of ideas which arises from certain inverse problems in geometry and dynamical systems. The main unifying theme is the study of the X-ray transform for compact simple manifolds with boundary and for closed manifolds carrying suitable Anosov flows. There are surprising similarities between these two cases and several of these questions can be dealt with a remarkable energy-type identity called the Pestov identity.

The spirit and content of the notes have considerable overlap with Sharafutdinov's book [**Sha94**] and in particular with his lecture notes *'Ray Transform on Riemannian manifolds. Eight lectures on Integral Geometry'* [**Sha**]. However, there are several differences. We focus almost exclusively on surfaces and thus we avoid the machinery of semi-basic tensor fields. Our treatment is purely based on the elementary Cartan structural equations on the unit circle bundle and is inspired by Uhlmann and Sharafutdinov's beautiful paper [**SU00**]. We feel that this makes first contact with the subject more accessible to students, while still providing them with the main underlying ideas. The notes also contain elementary treatments of magnetic flows and thermostats and several applications which are not covered in the references above. These include a thorough study of the regularity of the Anosov splitting, entropy production and transparent connections. It is also worth pointing out that the monograph by Knieper [**Kni02**] contains a more intrinsic derivation of the Pestov identity (in any dimension) as well as a comprehensive discussion of rigidity issues for geodesic flows.

CHAPTER 1

Introduction

We being by introducing the main players in this monograph. The first section defines the geodesic flow and explains the Anosov condition. The second section defines the X-ray transform, and presents a result on the kernel of the X-ray transform on the round sphere S^2. The final section is a brief introduction to contact and symplectic geometry, summarizing for the convenience of the reader the necessary background information for future chapters.

We will make the following assumption throughout in order to avoid needless repetition:

All manifolds in this book are connected and orientable, and all flows are of class C^∞ and without fixed points.

Occasionally this will be unnecessary, but for the sake of a uniform presentation we will keep these hypotheses throughout.

1.1. The geodesic flow and the Anosov condition

Let (M, g) be a closed (i.e. compact and without boundary) Riemannian manifold. Recall that a curve γ is called a *geodesic* if it is a solution to the 2nd order ODE called the *standard geodesic equation*

$$\frac{D\dot{\gamma}}{dt} = 0, \tag{1.1.1}$$

where $\frac{D}{dt}$ is the covariant derivative arising from the Levi-Civita connection on M. In general we will parametrize geodesics so they have unit speed, that is, $|\dot{\gamma}(t)| = 1$ for all $t \in \mathbb{R}$. A *closed geodesic* γ is a geodesic $\gamma : \mathbb{R} \to M$ such that there exists $T > 0$ such that $\gamma(0) = \gamma(T)$, and $\dot{\gamma}(0) = \dot{\gamma}(T)$ (note this is not the same as simply requiring the geodesic to be a loop). The smallest such value of T is called the *period* of γ.

Given $(x, v) \in TM$ there exists a unique geodesic $\gamma_{(x,v)}$ on M such that $\gamma_{(x,v)}(0) = x$ and $\dot{\gamma}_{(x,v)}(0) = v$ (in general we say a curve $\gamma : (-\epsilon, \epsilon) \to M$ is *adapted* to (x, v) if $\gamma(0) = x$ and $\dot{\gamma}(0) = v$). Since M is closed each geodesic is defined on all of $\mathbb{R}$. Given $t \in \mathbb{R}$, we define a diffeomorphism $\phi_t : TM \to TM$ as follows:

$$\phi_t(x, v) := (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)).$$

Note that ϕ_t is a *flow*, that is, $\phi_t \circ \phi_s = \phi_{t+s}$.

DEFINITION 1.1. We define the *unit sphere bundle SM* to be the fibre bundle over M given by

$$SM := \{(x, v) \in TM \ : \ |v| = 1\}.$$

Note that SM is a *sphere bundle*, since if $\pi : SM \to M$ denotes the footpoint map, we have $\pi^{-1}(x) \cong S^{n-1}$. SM is a manifold of dimension $2n - 1$, where $\dim M = n$. If M is closed then SM is closed.

Since a geodesic is parametrized to have unit speed, ϕ_t leaves SM invariant and thus ϕ_t defines a flow on SM, that is, if $(x, v) \in SM$ then $\phi_t(x, v) \in SM$. We call the restriction of ϕ_t to SM the *geodesic flow* of (M, g).

Let $\phi_t : N \to N$ be an arbitrary (smooth) flow on a closed Riemannian manifold (N, g). ϕ_t determines a vector field X, its *infinitesimal generator* defined by

$$X(x) = \frac{d}{dt}\Big|_{t=0} \phi_t(x). \tag{1.1.2}$$

In the case of the geodesic flow, X is a vector field on SM, known as the *geodesic vector field*. We say that $\phi_t : N \to N$ is a *nonsingular* flow if the infinitesimal generator X is nonsingular, that is, $X(x) \neq 0$ for all $x \in N$.

DEFINITION 1.2. A closed set $\Lambda \subseteq N$ is called *ϕ_t-invariant* (or simply 'invariant') if $\phi_t(\Lambda) \subseteq \Lambda$ for all $t \in \mathbb{R}$. A closed invariant set $\Lambda \subseteq N$ is called *hyperbolic* if there exist subbundles $E^s, E^u \subseteq T_\Lambda N$ (here $T_\Lambda N := \{(x, v) \in TN : x \in \Lambda\}$ - note this is not the same as $T\Lambda$) such that: for all $x \in \Lambda$

$$T_x N = \mathbb{R}X(x) \oplus E^s(x) \oplus E^u(x),$$

and for all $t \in \mathbb{R}$,

$$d\phi_t(E^s(x)) \subseteq E^s(\phi_t x),$$

$$d\phi_t(E^u(x)) \subseteq E^u(\phi_t x),$$

and for all $t \geq 0$,

$$\|d\phi_t|_{E^s}\| \leq Ce^{-\mu t},$$

$$\|d\phi_{-t}|_{E^u}\| \leq Ce^{-\mu t},$$

where $C, \mu > 0$ are constants.

If N is itself a hyperbolic set we say that ϕ_t is an *Anosov flow*. We call the subbundles E^s and E^u the *stable* and *unstable* bundles respectively.

REMARK 1.3. Actually by slightly decreasing the value of μ we can always assume that $C = 1$. That is, we can find a new metric g' such that the the corresponding constant $C(g') = 1$. Moreover, with this metric the subbundles $\mathbb{R}X$, E^s and E^u are orthogonal. We call the metric g' an *adjusted metric* on N. This is a common trick in hyperbolic dynamical systems and often simplifies many arguments. For completeness, here is a proof of this fact.

LEMMA 1.4. *Suppose $\phi_t : N \to N$ is an Anosov flow. Let g be a Riemannian metric on N, and let $C = C(g), \mu = \mu(g) > 0$ be constants such that for all $x \in N$ and all $t \geq 0$,*

$$|d_x\phi_t(v)| \leq Ce^{-\mu t}|v| \quad \text{for all } v \in E^s(x),$$

$$|d_x\phi_{-t}(v)| \leq Ce^{-\mu t}|v| \quad \text{for all } v \in E^u(x).$$

Then there exists a metric g' which is equivalent to g such that if C', μ' denote the corresponding constants for g' then $C' = 1$.

PROOF. Fix some $\mu' < \mu$. For $u \in \mathbb{R}X$, $v \in E^s$, $w \in E^u$ and large $T > 0$ set:

$$|u|' := |u|;$$

$$|v|' := \int_0^T e^{\mu' s}|d\phi_s(v)|\,ds;$$

$$|w|' := \int_0^T e^{\mu' s}|d\phi_{-s}(w)|\,ds;$$

$$|u + v + w|' := \sqrt{|u|^2 + |v|^2 + |w|^2}.$$

EXERCISE 1.5. Show that (for $T > 0$ large enough) the metric g' obtained form the norm $|\cdot|'$ satisfies the conditions of the lemma.

□

It is not hard to see that the subbundles E^s and E^u are necessarily continuous (see Exercise 11.1 in Chapter 11). Actually the distributions are more regular than this; we will discuss this much more fully in Chapter 11. Observe that the adjusted metric g' is as regular as the subbundles E^s and E^u.

The following theorem, which we will prove in Chapter 5 shows that Anosov flows appear rather frequently.

THEOREM 1.6. *If (M, g) is a closed surface with negative curvature then the geodesic flow on M is Anosov.*

REMARK 1.7. **A caveat on notation:** Unfortunately there is no generally accepted convention in the literature when it comes to naming the (un)stable bundles. The bundles E^s and E^u that we defined above are often referred to as the *strong* (un)stable bundles; this is to distinguish them from the *weak* (un)stable bundles that we will introduce later (see Exercise 3.24). As such, it is quite common to see the notation E^{ss} and E^{su} (for 'strong stable' and 'strong unstable') for what we call E^s and E^u. We call the weak stable bundle E^- and the weak unstable bundle E^+; sometimes (when the strong stable bundles are called E^{ss} and E^{su}) these bundles are called E^s and E^u. Even more confusingly, it is also not unusual to see the weak stable bundle referred to as E^+ and the weak unstable bundle referred to as E^- - the opposite of what we use. The conventions we have chosen to follow are probably the most common, although the reader however is *strongly cautioned* to make doubly sure whenever he or she reads papers in this area that they are fully aware of exactly which permutation of the above symbols the author of that particular paper is using!

1.2. The X-ray transform

DEFINITION 1.8. Let (M, g) be a closed manifold. Let $\mathcal{G}(M, g)$ denote the set of all closed geodesics on M. Let $h \in C^\infty(M, \mathbb{R})$. Consider $I[h] : \mathcal{G}(M, g) \to \mathbb{R}$ defined by

$$I\,[h]\,(\gamma) := \int_0^T h(\dot{\gamma}(t))dt, \tag{1.2.1}$$

where T is the period of γ. More generally, this defines a map $I : C^\infty(M, \mathbb{R}) \to \mathrm{Maps}(\mathcal{G}(M, g), \mathbb{R})$ called the *X-ray transform*. The main question we wish to answer in these notes is: can we reconstruct h from knowledge of $I[h]$? Since I is linear we could alternatively ask: what is the kernel of I?

We can ask a similar question for 1-forms: for $\theta \in \Omega^1(M)$ and $\gamma \in \mathcal{G}(M, g)$ define

$$I[\theta](\gamma) := \int_\gamma \theta$$

and then define $I : \Omega^1(M) \to \mathrm{Maps}(\mathcal{G}(M, g), \mathbb{R})$. Similarly we can define

$$I : \Gamma(\mathrm{Sym}^2(T^*M)) \to \mathrm{Maps}(\mathcal{G}(M, g), \mathbb{R})$$

by setting

$$I[\beta](\gamma) = \int_0^T \beta_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))dt$$

for a symmetric 2-tensor β.

For 1-forms the kernel of I is never trivial, since one easily sees that if θ is exact then $I\,[\theta] = 0$. In this case the interesting question becomes whether this natural obstruction is in fact the only obstruction. The following is a special case of one of the main results of this monograph, proved in Chapter 8.

THEOREM 1.9. *Let (M, g) be a closed surface and suppose the geodesic flow ϕ_t on M is Anosov. If $h \in C^\infty(M, \mathbb{R})$ and if $\theta \in \Omega^1(M)$ then*

$$I\,[h + \theta] = 0 \quad \Leftrightarrow \quad h \equiv 0 \text{ and } \theta \text{ is exact.}$$

Recall that the *Laplacian* of g is the map $\Delta_g : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$, defined by $\Delta_g(h) = \mathrm{div}(\mathrm{grad}\, h)$. Alternatively, in local coordinates,

$$\Delta_g(h) = \frac{1}{\sqrt{\det\, g_{ij}}}\partial_i\left(\sqrt{\det\, g_{ij}}\,\partial^i h\right), \qquad \partial^i h := g^{ij}\partial_j h.$$

We say that a metric g has *simple length spectrum* if all the closed geodesics are *nondegenerate* and no two have the same length, where by nondegenerate we mean the following: if E is the energy functional given by

$$E(\gamma) := \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2\, dt,$$

then the geodesics are precisely the critical points of E, and we say a geodesic is nondegenerate if it is nondegenerate in the sense of Morse theory as a critical point of E. It can be shown that the property of a metric having simple length spectrum is generic [**Abr70, Ano82**]. Let us denote by $\mathrm{Spec}(\Delta_g)$ the *spectrum of the Laplacian*, that is, the sequence $(\lambda_i) \subseteq \mathbb{R}$ of real numbers (counted with multiplicites) such that there exists $0 \neq h_i \in C^\infty(M,\mathbb{R})$ with $\Delta_g(h_i) = \lambda_i h_i$. The following result is due to Guillemin [**Gui78**] and Duistermaat and Guillemin [**DG75**], and concerns the so-called *Schrödinger operators*

$$h \in C^\infty(M,\mathbb{R}) \mapsto \Delta_g(h) + qh \in C^\infty(M,\mathbb{R}),$$

where $q \in C^\infty(M,\mathbb{R})$ is a fixed smooth function.

THEOREM 1.10. *Let (M,g) be a closed manifold such that g has simple length spectrum. Suppose $q_1, q_2 \in C^\infty(M,\mathbb{R})$. Then if* $\mathrm{Spec}(\Delta_g + q_1) = \mathrm{Spec}(\Delta_g + q_2)$ *it holds that* $I[q_1] = I[q_2]$.

In fact, as pointed out by Guillemin and Kazhdan [**GK80**], in the case of negatively curved surfaces by combining Theorem 1.6 and Theorem 1.9 we can say more. Namely, we have the following *spectral rigidity* result:

COROLLARY 1.11. *Let (M,g) be a closed negatively curved surface with simple length spectrum. Suppose $q_1, q_2 \in C^\infty(M,\mathbb{R})$. Then:*

$$\mathrm{Spec}(\Delta_g + q_1) = \mathrm{Spec}(\Delta_g + q_2) \quad \Rightarrow q_1 = q_2.$$

We shall come back to the issue of spectral rigidity in Chapter 12 (see Section 12.2).

It is not true in general that I acting on functions has zero kernel. Consider the case of S^2, with the usual metric of constant curvature 1. Here geodesics are great circles, and they are all closed with period 2π (we call manifolds all of whose geodesics are closed *Zoll manifolds*). A great circle can be identified with a point on S^2: the correspondence associates the geodesic traveling counter-clockwise through the equator with the north pole $N = (0,0,1)$. Thus we may identify $\mathscr{G}(S^2)$ with S^2 and consider I as a map $C^\infty(S^2) \to C^\infty(S^2)$, defined by

$$I\,[h]\,(x) = \int_0^{2\pi} h(\gamma(t))dt, \qquad x \leftrightarrow \gamma.$$

EXERCISE 1.12. Show that if h is an odd function then $I\,[h] = 0$.

We have a decomposition

$$C^\infty(S^2) = C^\infty_{\mathrm{odd}}(S^2) \oplus C^\infty_{\mathrm{even}}(S^2),$$

and the exercise asserts that $C^\infty_{\mathrm{odd}}(S^2) \subseteq \ker I$. In fact, the following theorem holds.

THEOREM 1.13. *The kernel of the X-ray transform I on S^2 with its standard metric of constant curvature* 1 *is precisely the odd functions on S^2:*

$$\ker I = C^\infty_{\mathrm{odd}}(S^2).$$

Moreover $I : C^\infty_{\mathrm{even}}(S^2) \to C^\infty_{\mathrm{even}}(S^2)$ is bijective.

We conclude this opening section by presenting a proof of this theorem, following [**Gui76**, Appendix A]. The proof will use some representation theory and Fourier analysis. These methods will not reappear in the rest of the book (although we do some Fourier analysis in Chapter 13); the reader who wishes to get to the core of the material quicker is invited to skip ahead to the next section.

We will need the following standard formula for the Laplacian $\Delta_{S^{n-1}}$ acting on functions on S^{n-1}. Given $f \in C^\infty(\mathbb{R}^n)$, let $\bar{f}$ denote $f|_{S^{n-1}}$. We first quote the following result relating $\Delta_{\mathbb{R}^n}$ and $\Delta_{S^{n-1}}$; its proof can be found in [**GHL04**, Proposition 4.48]:

$$\overline{\Delta_{\mathbb{R}^n}(f)} = \Delta_{S^{n-1}}(\bar{f}) + \overline{\frac{\partial^2 f}{\partial r^2}} + (n-1)\overline{\frac{\partial f}{\partial r}}, \tag{1.2.2}$$

where r is the radial coordinate.

Let

$$\mathbf{P}^n_k := \{\text{homogeneous polynomials of degree } k \text{ on } \mathbb{R}^n\}$$

and

$$\mathbf{H}^n_k := \left\{P \in \mathbf{P}^n_k \,:\, \Delta_{\mathbb{R}^n}(P) = 0\right\} \subseteq \mathbf{P}^n_k$$

denote the *harmonic* homogeneous polynomials of degree k on $\mathbb{R}^n$.

We write $P \in \mathbf{P}^n_k$ as

$$P = r^k \bar{P},$$

and hence for $P \in \mathbf{P}_k^n$, (1.2.2) reduces to

$$\overline{\Delta_{\mathbb{R}^n}(P)} = \Delta_{S^{n-1}}(\bar{P}) + k(k+n-2)\bar{P},$$

and so if $P \in \mathbf{H}_k^n$ then

$$\Delta_{S^{n-1}}(\bar{P}) = -k(k+n-2)\bar{P},$$

so that $\bar{P}$ is an eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue $-k(k+n-2)$. Write $\overline{\mathbf{P}}_k^n := \{\bar{P} \,:\, P \in \mathbf{P}_k^n\}$ and similarly define $\overline{\mathbf{H}}_k^n := \{\bar{P} \,:\, P \in \mathbf{H}_k^n\}$.

We briefly describe the representation theory we need on $SO(n)$. We define an action of $O(n)$ on $\overline{\mathbf{P}}_k^n$ by setting

$$(g \cdot \bar{P})(x) := \bar{P}(g^{-1}x)$$

for $P \in \overline{\mathbf{P}}_k^n$ and $g \in O(n)$.

EXERCISE 1.14. Show that the Laplacian commutes with this action, that is,

$$\Delta(g \cdot P) = g \cdot \Delta(P),$$

and hence this action descends to give an action on $\overline{\mathbf{H}}_k^n$.

The following theorem is standard (see for instance, [**Sep07**, Theorem 2.33]).

THEOREM 1.15. *$\overline{\mathbf{H}}_k^n$ is an irreducible $O(n)$-module and for $n \geq 3$ is also an irreducible $SO(n)$-module.*

Moreover $L^2(S^n)$ decomposes as the Hilbert space direct sum

$$L^2(S^n) = \bigoplus_{k=0}^{\infty} \overline{\mathbf{H}}_k^n.$$

We now restrict to the case $n = 3$, and write $\mathbf{P}_k$ for $\mathbf{P}_k^3$ etc. The key observation we need is that the X-ray transform I *commutes* with the action of $SO(3)$ on S^2. It then follows immediately from *Schur's Lemma* (see [**Sep07**, Theorem 2.12]) that I maps $\overline{\mathbf{H}}_k$ into itself and there exist constants $\{c_k\} \subseteq \mathbb{R}$ such that

$$I|_{\overline{\mathbf{H}}_k} = c_k \cdot \mathrm{Id}.$$

As we observed earlier, clearly $c_{2k+1} = 0$ for all $k \in \mathbb{N}$, since $\overline{\mathbf{H}}_{2k+1} \subseteq C^\infty_{\mathrm{odd}}(S^2)$.

PROPOSITION 1.16. *For $k \in \mathbb{N}$,*

$$c_{2k} = (-1)^{2k} \int_0^{2\pi} (\cos\theta)^{2k}\, d\theta \tag{1.2.3}$$

$$= 2\pi(-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots 2k}. \tag{1.2.4}$$

PROOF. We take advantage of the fact that we need only check the result on a fixed $P \in \mathbf{H}_{2k}$ of our choice and a fixed point in S^2.

Consider

$$P(x,y,z) := \sum_{i=0}^{2k} a_i x^{2k-i} z^i,$$

for some constants $a_i \in \mathbb{R}$. Of course the a_i cannot be arbitrary; the assumption P is harmonic implies

$$\begin{aligned} 0 &= \Delta_{\mathbb{R}^n}(P) \\ &= \sum_{i=0}^{2k-2} a_i(2k-i)(2k-i-1)x^{2k-i-2}z^i + \sum_{i=2}^{2k} a_i i(i-1)x^{2k-i}z^{i-2} \\ &= \sum_{i=2}^{2k-2} \{a_{i-2}(2k-i+2)(2k-i+1) + a_i i(i-1)\}\, x^{2k-i}z^{i-2}, \end{aligned}$$

and hence

$$\frac{a_i}{a_{i-2}} = -\frac{(2k-i+2)(2k-i+1)}{i(i-1)},$$

and so

$$\frac{a_{2k}}{a_0} = (-1)^k \frac{2k(2k-1)\cdots 2\cdot 1}{1\cdot 2\cdot 3\cdots(2k-1)2k} = (-1)^k. \tag{1.2.5}$$

Then if $\gamma : [0, 2\pi] \to S^2$ denotes the geodesic that goes once round the equator of S^2, so γ corresponds to the North pole $N = (1, 0, 0)$ on S^2 then

$$\begin{aligned} I[\bar{P}](N) &= \int_0^{2\pi} P(\gamma(t))dt \\ &= \int_0^{2\pi} P(\cos t, \sin t, 0)dt \\ &= a_0 \int_0^{2\pi} (\cos t)^{2k} dt. \end{aligned}$$

But also by assumption

$$\begin{aligned} I[\bar{P}](N) &= c_{2k} P(N) \\ &= c_{2k} a_{2k}. \end{aligned}$$

Thus we conclude that

$$c_{2k} = \frac{a_0}{a_{2k}} \int_0^{2\pi} (\cos t)^{2k} dt,$$

and thus by (1.2.5), we obtain

$$c_{2k} = (-1)^k \int_0^{2\pi} (\cos t)^{2k} dt.$$

We leave the verification that the integral (1.2.3) does indeed give the product (1.2.4) as stated as an exercise for the reader. □

This immediately proves that the kernel of I is precisely the odd functions; namely if $I[f] = 0$ then expanding f into harmonic polynomials and using the fact that $c_{2k} \neq 0$ for all k shows $f \in C^\infty_{\text{odd}}(S^2)$, that is, $\ker I \subseteq C^\infty_{\text{odd}}(S^2)$, and we have already observed the reverse inclusion trivially holds. It will take a bit more work however to prove the second assertion of Theorem 1.13. We shall really only sketch the ideas involved.

Let

$$h(s) := \int_0^{2\pi} (\cos t)^{s/2} dt,$$

so that

$$c_{2k} = (-1)^k h(2k).$$

The essence of the proof is to show that I is a *smoothing operator* of order $-\frac{1}{2}$. More precisely, we will show that I maps the Sobolev space $H^s_{\text{even}}(S^2)$ into $H^{s+1/2}_{\text{even}}(S^2)$. There are various ways to define the space $H^s(S^2)$; informally one thinks of a function $f : S^2 \to \mathbb{R}$ lying in $H^s(S^2)$ for $s \in \mathbb{N}$ if it has s derivatives in L^2 (see for instance [**Wel08**, Chapter IV, Section 1] for more information) . For our purposes it suffices to observe that

$$\overline{\mathbf{H}}_k \subseteq H^s(S^2) \quad \text{for all} \quad s \in \mathbb{R},\ k \in \mathbb{N}$$

and we may define the H^s norm $\|\cdot\|_s$ on $\overline{\mathbf{H}}_k$ by

$$\|\bar{P}\|_s := k^s \|\bar{P}\|_{L^2} \quad \text{for} \quad P \in \mathbf{H}_k.$$

PROOF. *(of Theorem 1.13)*

We now undertake an analysis of the asymptotic behavior of $h(s)$; using the product expansion (1.2.4) together with Wallis's formula

$$\sqrt{\pi} = \lim_{k\to\infty} \frac{1}{\sqrt{k}} \cdot \frac{2\cdot 4\cdot 6\cdots 2k}{1\cdot 3\cdot 5\cdots(2k-1)}$$

we discover

$$c_{2k} \sim (-1)^k \sqrt{\frac{4\pi}{k}},$$

and hence for $s \in \mathbb{N}$ large,

$$h(s) = \sqrt{2\pi}\cdot s^{-1/2} + O(s^{-1/2}).$$

It follows that we may choose a constant C independent of s such that if $f \in \bigoplus_{k=0}^{\infty} \overline{\mathbf{H}}_{2k}$ then

$$C^{-1} \|f\|_s \leq \|I[f]\|_{s+1/2} \leq C \|f\|_s,$$

and hence I maps the Sobolev space $H^s_{\text{even}}(S^2)$ bijectively onto $H^{s+1/2}_{\text{even}}(S^2)$.

Since s was arbitrary, it follows that I is bijective on $C^\infty_{\text{even}}(S^2)$ and this completes the proof. □

EXERCISE 1.17. Consider the X-ray transform $I : \Omega^1(S^2) \to C^\infty(S^2)$ on 1-forms on S^2 and let $\sigma : S^2 \to S^2$ be the antipodal map. A 1-form θ is said to be *odd* if $\sigma^*\theta = -\theta$ and *even* if $\sigma^*\theta = \theta$. Show that any odd form is in the kernel of I. Moreover, show that an even form is in the kernel of I if and only if it is exact (see [**Mic78**, Section 8] if you get stuck).

1.3. A very brief introduction to contact and symplectic geometry

In this section we will give a very brief introduction to contact and symplectic geometry; in particular we will show that SM admits the structure of a contact manifold, whose Reeb vector field is precisely the geodesic vector field. Two references for this section are [**Gei08**, Section 1.5], and [**Pat99**, Chapter 1], where most of this material has come from.

DEFINITION 1.18. Let M^n be a smooth manifold, and let $\tau : T^*M \to M$ be the footpoint map. Define a differential 1-form λ on T^*M, called the *Liouville form* by

$$\lambda_{(x,p)}(\xi) := p(d_{(x,p)}\tau(\xi)), \quad (x,p) \in T^*M, \quad \xi \in T_{(x,p)}T^*M.$$

If $\left(x^1, \dots, x^n, p_1, \dots, p_n\right)$ are local coordinates on T^*M we claim

$$\lambda = p_i dx^i.$$

Denote the induced fibre coordinates on TT^*M by $\left(\dot{x}^i, \dot{p}_i\right)$, so that elements of TT^*M are locally written as $\xi = \dot{x}^i \frac{\partial}{\partial x^i} + \dot{p}_i \frac{\partial}{\partial p_i}$. Then we have

$$\begin{aligned} \lambda(\xi) &= \lambda\left(\dot{x}^i \frac{\partial}{\partial x^i} + \dot{p}_i \frac{\partial}{\partial p_i}\right) \\ &= p_j dx^j \left(d\tau\left(\dot{x}^i \frac{\partial}{\partial x^i} + \dot{p}_i \frac{\partial}{\partial p_i}\right)\right) \\ &= p_j dx^j \left(\dot{x}_i \frac{\partial}{\partial x^i}\right) \\ &= p_i \dot{x}^i \\ &= p_j dx^j \left(\dot{x}^i \frac{\partial}{\partial x^i} + \dot{p}_i \frac{\partial}{\partial p_i}\right) \\ &= p_i dx^i(\xi). \end{aligned}$$

DEFINITION 1.19. A *symplectic form* on a smooth manifold N^{2n} is a differential 2-form ω that is closed and nondegenerate, that is, ω^n is a nowhere vanishing $2n$-form. The pair (N, ω) is then called a *symplectic manifold*.

Going back to the situation above, the 2-form $\omega = -d\lambda = dx^i \wedge dp_i$ is clearly closed, and since

$$\frac{(-1)^{n-1}}{n!}\omega^n = dx^1 \wedge \cdots \wedge dx^n \wedge dp_1 \wedge \cdots \wedge dp_n \neq 0,$$

we see that (T^*M, ω) is a symplectic manifold. We call ω the *canonical symplectic form* on T^*M.

DEFINITION 1.20. A *contact manifold* (N, α) is a smooth manifold N^{2n-1} equipped with a *contact form* α, that is, a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on N. Associated to a contact form α is a unique *Reeb vector field* R defined by

$$\alpha(R) = 1, \quad i_R d\alpha = 0.$$

Indeed, for each point $x \in N$, $d\alpha_x$ is a skew-symmetric 2-form of maximal rank $2n-2$, and hence has a 1-dimensional kernel, which then defines $R(x)$ up to a scalar, that is, this defines an oriented line field $\mathbb{R}R \subseteq TN$. The contact condition implies α is non-trivial on this line field, and hence there exists a unique smooth non-vanishing section satisfying the normalization condition $\alpha(R) = 1$. Strictly speaking, what we have just defined is a *coorientable* contact manifold. The word 'coorientable' refers to the fact that $\mathbb{R}R$ is an *oriented* line field.

The aim of the rest of this section is to introduce a 1-form α on SM such that α is a contact form, and such that the associated Reeb vector field of α is precisely the geodesic vector field X. We will take a somewhat long winded approach to this, by first using the Liouville form to make S^*M into a contact manifold, and then using the musical isomorphism to transfer this contact structure to SM.

DEFINITION 1.21. Let (N^{2n}, ω) be a symplectic manifold. A *Liouville vector field* Y on N is a vector field satisfying

$$L_Y \omega = \omega.$$

EXERCISE 1.22. Show that if Y is a Liouville vector field then the 1-form $\alpha := i_Y \omega$ is a contact form on any hypersurface on N transverse to Y.

Let us go back to the case of (T^*M, ω) where ω is the canonical symplectic form. We can define a unique vector field Y on T^*M by $i_Y \omega = -\lambda$. Then since $d(i_Y \omega) = -d\lambda = \omega$, Y is a Liouville vector field. Then we claim that Y is transverse to S^*M. Indeed, the integral curves of Y through $(x, p) \in T^*M$ can be written as $\gamma(t) = (x, e^t p)$ and hence for $p \neq 0$, we have

$$\frac{d}{dt} \langle \gamma(t), \gamma(t) \rangle = \frac{d}{dt} e^{2t} \langle p, p \rangle > 0,$$

where here $\langle \cdot, \cdot \rangle$ denotes the induced metric on T^*M. Hence by Exercise 1.22 we conclude that $\lambda = -i_Y \omega$ induces a contact form on S^*M.

We wish to compute the Reeb vector field R of λ. In fact, we prove that $R = \hat{g}X$ where X is the geodesic vector field, and $\hat{g} : TM \to T^*M$ is the so-called 'musical isomorphism' determined by the Riemannian metric (see [**Pat99**, Definition 1.36]). In coordinates (x^i) on M,

$$\hat{g} : v^i \frac{\partial}{\partial x^i} \mapsto g_{ij} v^i dx^j$$

with inverse

$$\hat{g}^{-1} : p_j dx^j \mapsto g^{ij} p_j \frac{\partial}{\partial x^i}.$$

If the (x^i) are *normal* coordinates at x then $g_{ij}(x) = \delta_{ij}$ and $\partial_k g_{ij}(x) = 0$, and hence the musical isomorphism $\hat{g}$ is the 'identity' at x, i.e.

$$g_{ij} v^i = v^j.$$

Let the corresponding local coordinates on TM and T^*M be denoted by $(x^1, \dots, x^n, v^1, \dots, v^n)$ and $(x^1, \dots, x^n, p_1, \dots, p_n)$ respectively. Moreover if $(\dot{x}^i, \dot{v}^i)$ and $(\dot{x}^i, \dot{p}_i)$ denote the induced fibre coordinates on TTM and TT^*M then the map $d\hat{g}$ is also the identity on the fibres over x.

In normal coordinates (x^i) at $x \in M$ the local coordinate description $(\dot{x}^i, \dot{v}^i)$ of the integral curves of X running through $(x, v) \in SM$ is

$$t \mapsto (tv^i, v^i).$$

Hence if $v = v^i \frac{\partial}{\partial x^i}|_x \in T_x M$ then

$$X(x, v) = v^i \frac{\partial}{\partial x^i}\Big|_{(x,v)} + 0 \cdot \frac{\partial}{\partial v^i}\Big|_{(x,v)}.$$

Since $d\hat{g}$ is the identity map on the fibres over x, we have

$$(\hat{g}X)(x, v) = v^i \frac{\partial}{\partial x^i}\Big|_{(x,\hat{g}v)} + 0 \cdot \frac{\partial}{\partial p_i}\Big|_{(x,\hat{g}v)}.$$

Thus if $(x, v) \in SM$,

$$\begin{aligned} \lambda_{(x,\hat{g}v)}((\hat{g}X)(x, v)) &= v^i dx^i|_x \left(v^i \frac{\partial}{\partial x^i}\Big|_x \right) \\ &= \sum_i (v^i)^2 \\ &= 1. \end{aligned}$$

Moreover

$$\begin{aligned} d\lambda_{(x,\hat{g}v)}((\hat{g}X)(x,v),\cdot) &= dp_i \wedge dx^i((\hat{g}X)(x,v),\cdot) \\ &= -v^i dp_i|_{(x,\hat{g}v)}(\cdot). \\ &= 0, \end{aligned}$$

since $\sum_i p_i dp_i = 0$ on $T_{(x,p)}S^*M$, which follows by differentiating $g^{ij} p_i p_j = 1$ and using the fact that $g_{ij}(x) = \delta_{ij}$ and $\partial_k g_{ij}(x) = 0$.

Thus we have proved that the Reeb vector field R of λ is $\hat{g}X$.

DEFINITION 1.23. We can now give SM the structure of a contact manifold. Consider the 1-form $\alpha = \hat{g}^*\lambda$ on TM. Let $(x, v) \in SM$. Then

$$\begin{aligned} \alpha_{(x,v)}(\xi) &= \lambda_{(x,\hat{g}v)}(d\hat{g}(\xi)) \\ &= (\hat{g}v)(d_{(x,\hat{g}v)}\tau \circ d\hat{g}(\xi)) \\ &= (\hat{g}v)(d_{(x,\hat{g}v)}(\tau \circ \hat{g})(\xi)) \\ &= (\hat{g}v)(d\pi(\xi)) \\ &= \langle v, d\pi(\xi)\rangle . \end{aligned}$$

Since λ is a contact form on S^*M, it follows α is a contact form on SM. Moreover the Reeb vector field of α is just X, since we showed $\hat{g}X = R$. In fact there is another way to relate α and X; in Chapter 4 we will introduce a metric $\langle\langle\cdot,\cdot\rangle\rangle$ on TM called the *Sasaki metric* (see 4.2) . Under this metric α and X will be dual to each other, that is,

$$\alpha(\xi) = \langle\langle X, \xi\rangle\rangle .$$

The final result we need (we will use this in Chapter 7) is the following.

Let $H : TM \to \mathbb{R}$ denote the *'kinetic energy' Hamiltonian* defined by

$$H(x, v) = \frac{1}{2}|v|^2 .$$

Then on the entire tangent bundle TM,

$$i_X d\alpha = -dH.$$

Note that this is in agreement with what we have already shown, since on SM, $dH = 0$ as H is constant on SM.

EXERCISE 1.24. Prove the lemma (the reader may find a proof using local coordinates in [**GHL04**, Theorem 2.124], or an intrinsic proof is given in [**Pat99**, Proposition 1.21], which makes use of the result of Exercise 4.3 below).

CHAPTER 2

The boundary rigidity problem

In this chapter we will survey in some depth what at first sight is a seemingly unrelated problem; that of boundary rigidity. We shall see however that the infinitesimal version of the boundary rigidity problem can be solved in certain special cases by studying the kernel of the relevant X-ray transform. In the penultimate chapter of this book we will return to this circle of ideas.

2.1. Introduction to the problem

Let $(M, \partial M, g)$ be a compact Riemannian manifold with boundary. The *boundary rigidity problem* is the following: recall we define the *geodesic distance function* on M to be

$$d_g : M \times M \to \mathbb{R}, \qquad d_g(x, y) := \inf_{\gamma \in \Omega_{x,y}} \ell_g(\gamma), \tag{2.1.1}$$

where $\Omega_{x,y}$ denotes the set of smooth curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$, and

$$\ell_g(\gamma) := \int_0^1 |\dot{\gamma}(t)|\, dt.$$

Suppose that the following is known:

$$d_g(x, y), \quad \forall x, y \in \partial M.$$

Can we reconstruct g on the interior of M from this information?

Here is the motivation for this problem. Suppose M is a ball $\{x \in \mathbb{R}^3 : |x| \le R\}$ (and so ∂M is the sphere $\{x \in \mathbb{R}^3 : |x| = R\}$) equipped with a metric $ds^2 = f(r) \sum_{i=1}^3 (dx^i)^2$ where $r = |x|$ and $f(r)$ is a positive function of the radius (so the metric is *conformal* to the standard one). Physically, this is meant to represent a spherically symmetric model of the Earth with an index of refraction depending only on the radius. The boundary distance function corresponds to the travel times of e.g. seismic waves going through the Earth and measured at the surface. The problem goes back to Herglotz [**Her05**] and Wiechert and Zoeppritz [**WZ07**], who in the early part of the 20th century found a way to determine $f(r)$ from the restriction of d_g to $\partial M \times \partial M$.

In this generality the answer is easily seen to be no: if $\psi : M \to M$ is a diffeomorphism such that $\psi|_{\partial M} = \mathrm{Id}$, then if $g' := \psi^* g$ we clearly have $d_g = d_{g'}$ on $\partial M \times \partial M$, since if $\gamma \in \Omega_{x,y}$ then $\psi \circ \gamma \in \Omega_{x,y}$ and $\ell_{\psi^* g}(\gamma) = \ell_g(\gamma)$, but of course there is no reason for g to equal g' on the interior of M. Thus it is only sensible to frame the question as to whether knowledge of $d_g|_{\partial M \times \partial M}$ allows us to determine g on the interior of M up to such a diffeomorphism. We will say that g is *boundary rigid* if $d_g|_{\partial M \times \partial M}$ determines g up to such a diffeomorphism ψ.

Regrettably however the answer is still easily seen to be negative. Suppose M contains an open set U on which g is very large. Then all length minimizing curves will avoid U, and thus d_g will not carry any information about $g|_U$. Thus we can alter g on U (but keeping it large) and not affect d_g on $\partial M \times \partial M$. Here is an example: take M to be the upper hemisphere of S^2, and let g_0 denote the natural metric on M. Note that $d_{g_0}(x, y)$ for any two boundary points is realized as the length of the shortest arc on ∂M connecting x and y. Now take a non-negative function f supported on $U \subseteq M$ and let $g_1 = (1 + f)g_0$. Then $d_{g_0} = d_{g_1}$ on $\partial M \times \partial M$, but (M, g_0) and (M, g_1) are not isometric as since $\mathrm{Vol}(M, g_1) > \mathrm{Vol}(M, g_0)$.

We need to impose further conditions on $(M, \partial M, g)$ in order to have a chance of making progress. Consider the boundary ∂M of a Riemannian manifold M. Then for $x \in \partial M$,

$$T_x^{\perp} \partial M := \{v \in T_x M : \langle v, w \rangle = 0 \text{ for all } w \in T_x \partial M\}$$

is a 1-dimensional vector space, and thus there exist precisely two unit vectors in $T_x^{\perp}\partial M$. We say that a vector $v \in T_x M$ is called an *unit inward pointing normal* to ∂M at x if $v \in T_x^{\perp}\partial M$, $|v| = 1$ and choosing orientation-compatible local coordinates (x^i) at x such that $\partial M = \{x^n = 0\}$ then writing $v = v^i \partial_i$ we have $v^n > 0$. Similarly v is an *unit outward pointing normal* if $-v$ is an unit inward pointing normal. There exists a unique unit inward pointing normal vector field along ∂M, that is, a map $\nu : \partial M \to T_{\partial M} M$ such that $\nu(x)$ is the unit inward pointing normal at x.

We define the *second fundamental form* S_x of M at $x \in \partial M$ by

$$S_x : T_x\partial M \times T_x\partial M \to \mathbb{R}, \qquad S_x(u, w) := -\langle \nabla_u \nu, w\rangle .$$

We say that ∂M is *strictly convex* if S_x is positive definite for all $x \in \partial M$. Here is the crucial condition that we will impose.

DEFINITION 2.1. Let $(M, \partial M, g)$ be a compact manifold with boundary. We say that $(M, \partial M, g)$ is *simple* if ∂M is strictly convex and for all $x \in M$,

$$\exp_x : \exp_x^{-1}(M) \subset T_x M \to M$$

is a diffeomorphism. These two conditions imply that any two points in M are joined by a unique geodesic which depends smoothly on the endpoints, and that if $x, y \in M$ and γ is the unique geodesic from x to y then $\gamma^{-1}(\partial M) \subseteq \{x, y\} \cap \partial M$, that is, the interior of γ does not intersect ∂M.

EXERCISE 2.2. Show that any simple manifold is diffeomorphic to a ball in $\mathbb{R}^n$ (a proof may be found in [**Sha94**]). Show that the condition of being simple is C^2-open in the metric g.

CONJECTURE 2.3. *(*[**Mic81**]*) Simple manifolds are boundary rigid within the class of simple metrics.*

This is known to be true generically, and has been proved completely by Pestov and Uhlmann in dimension two - see [**PU05**].

2.2. Deformation boundary rigidity

Let us consider the linearization of the boundary rigidity problem, which is sometimes easier to solve. We refer the reader to [**Sha94**, Chapter 1], for a more thorough discussion of the differences between the two problems.

Let $\{g_s : s \in (-\epsilon, \epsilon)\}$ be a family of metrics such that $(M, \partial M, g_s)$ is simple for all s. Suppose that $d_s := d_{g_s}$ satisfies $d_s = d_0$ on $\partial M \times \partial M$ for all $s \in (-\epsilon, \epsilon)$. We say the family $\{g_s\}$ is *trivial* if there exists a smooth family $\{\psi_s\}$ of diffeomorphisms such that $\psi_0 = \mathrm{Id}$ and $\psi_s|\partial M = \mathrm{Id}$ and $g_s = \psi_s^* g_0$. We say $(M, \partial M, g)$ is *deformation boundary rigid* if any such family $\{g_s\}$ with $g_0 = g$ is trivial.

Take $x, y \in \partial M$. Let γ_s denote the unique geodesic from x to y under g_s (γ_s exists and is unique since $(M, \partial M, g_s)$ is simple), parametrized with speed 1. Since $d_s = d_0$ on $\partial M \times \partial M$, if $T := d_0(x, y)$ then all the γ_s are defined on $[0, T]$. Consider the energy functional

$$E_s(\gamma) := \int_a^b |\dot\gamma(t)|_s^2\, dt \quad \text{for } \gamma : [a, b] \to M.$$

Note that $E_s(\gamma_s) \equiv T$. We simplify the problem by linearizing. Namely, compute:

$$0 = \frac{d}{ds}\Big|_{s=0} T = \frac{d}{ds}\Big|_{s=0} E_s(\gamma_s) = \int_0^T \frac{\partial g_s}{\partial s}\Big|_{s=0}(\dot\gamma_0(t), \dot\gamma_0(t))dt + \frac{d}{ds}\Big|_{s=0} E_0(\gamma_s). \tag{2.2.1}$$

Now we can consider γ_s as a variation of γ_0, and since γ_0 is a critical point of E_0, we have $\frac{d}{ds}\Big|_{s=0} E_0(\gamma_s) = 0$, and thus writing

$$\beta := \frac{\partial g_s}{\partial s}\Big|_{s=0}, \tag{2.2.2}$$

β is a symmetric 2-tensor and (2.2.1) reduces to

$$I_{g_0}[\beta](\gamma_0) = \int_0^T \beta_{\gamma_0(t)}(\dot\gamma_0(t), \dot\gamma_0(t))dt = 0.$$

But x and y were arbitrary, and since every geodesic begins and ends on ∂M, we conclude that

$$I_{g_0}[\beta] = 0,$$

where here I_{g_0} is the X-ray transform defined on 2-tensors with respect to the metric g_0. In other words, we have proved:

LEMMA 2.4. *Let $(M, \partial M)$ be a compact manifold with boundary and $\{g_s\}$ a family of simple metrics on M such that $d_s = d_0$ on $\partial M \times \partial M$ for all s. Define the symmetric 2-tensor β by* (2.2.2)*. Then β is in the kernel of the X-ray transform with respect to g_0.*

DEFINITION 2.5. A symmetric 2-tensor β is called a *potential* 2-tensor if there exists a smooth vector field Z on M such that

$$\beta_x(v, w) = \langle \nabla_v Z(x), w \rangle + \langle v, \nabla_w Z(x) \rangle$$

for all $x \in M$ and $Z|_{\partial M} = 0$.

Now let $(M, \partial M)$ be a compact manifold with boundary and $\{g_s\}$ a trivial family of simple metrics on M, with associated family $\{\psi_s\}$ of diffeomorphisms. Let Z denote the vector field defined by

$$Z = \frac{d}{ds}\Big|_{s=0} \psi_s.$$

Note that $Z|_{\partial M} = 0$.

Then with β defined as in (2.2.2),

$$\beta(v, w) = \frac{\partial}{\partial s}\Big|_{s=0} g_0(d\psi_s(v), d\psi_s(w)). \tag{2.2.3}$$

Let $\nabla = \nabla_0$ denote the Levi-Civita connection of g_0. Thus if we take smooth curves $a : (-\delta, \delta) \to M$ adapted to (x, v) and $b : (-\delta', \delta') \to M$ adapted to (x, w) we obtain

$$\begin{aligned}
\beta_x(v, w) &= \frac{\partial}{\partial s}\Big|_{s=0} g_0(d\psi_s(v), d\psi_s(w)) \\
&= \frac{\partial}{\partial s}\Big|_{s=0} g_0 \left(\frac{\partial}{\partial t}\Big|_{t=0} \psi_s \circ a(t), \frac{\partial}{\partial t}\Big|_{t=0} \psi_s \circ b(t) \right) \\
&\overset{(*)}{=} \left\langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} \psi_s \circ a(t), w \right\rangle + \left\langle v, \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} \psi_s \circ b(t) \right\rangle \Big|_{s=t=0} \\
&\overset{(**)}{=} \left\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} \psi_s \circ a(t), w \right\rangle + \left\langle v, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} \psi_s \circ b(t) \right\rangle \Big|_{s=t=0} \\
&= \langle \nabla_v Z(x), w \rangle + \langle v, \nabla_w Z(x) \rangle ,
\end{aligned}$$

where $(*)$ is the statement that ∇ is compatible with the metric and $(**)$ is the statement that ∇ is symmetric. Thus we have proved that if $\{g_s\}$ is trivial then β is a potential tensor with $Z|_{\partial M} = 0$. Conversely we may repeat this argument backwards, by integrating the equations:

$$\frac{\partial \psi_s}{\partial s} = Z_s \circ \psi_s, \quad \psi_0 = \mathrm{Id},$$

to obtain the following:

PROPOSITION 2.6. *Let $(M, \partial M)$ be a compact manifold with boundary and $\{g_s\}$ a family of simple metrics on M such that $d_s = d_0$ on $\partial M \times \partial M$ for all s. Define a family of symmetric 2-tensors $\{\beta_s\}$ by*

$$\beta_s(v, w) = \frac{\partial g_s}{\partial s}(v, w).$$

Then $\{g_s\}$ is a trivial family if and only if $\{\beta_s\}$ is a family of potential 2-tensors with the corresponding vector fields Z_s depending smoothly on s.

EXERCISE 2.7. Complete the details for the converse direction in the proposition above.

In the Chapter 12 we will prove the following two-dimensional result (see Theorem 12.5).

THEOREM 2.8. *Let $(M, \partial M, g)$ be a simple compact surface with negative curvature. Let β be a symmetric 2-tensor. If $I\,[\beta] = 0$, then β is a potential 2-tensor.*

Combining this result with Lemma 2.4 and Proposition 2.6 we obtain immediately the following result.

COROLLARY 2.9. *Compact simple negatively curved Riemannian surfaces are deformation boundary rigid.*

In fact, the curvature assumption is not needed as has recently been proved by Sharafutdinov [**Sha07**].

2.3. Lens rigidity

DEFINITION 2.10. Let $(M, \partial M, g)$ be a simple manifold, and let $x \in \partial M$. Let $\nu(x)$ denote the unit inward pointing normal. Set

$$\partial_+(SM) := \{(x, v) \in SM \,:\, x \in \partial M,\ \langle v, \nu(x)\rangle \geq 0\},$$

$$\partial_-(SM) := \{(x, v) \in SM \,:\, x \in \partial M\ \langle v, \nu(x)\rangle \leq 0\}.$$

Then we have

$$\partial_+(SM) \cup \partial_-(SM) = \partial(SM),$$

$$\partial_+(SM) \cap \partial_-(SM) = S(\partial M).$$

Let also

$$TM_0 := \{(x, v) \in TM \,:\, v \neq 0\}.$$

Given $(x, v) \in TM_0$, there exists a unique geodesic $\gamma_{(x,v)}$ adapted to (x, v); moreover $\gamma_{(x,v)}$ is maximally defined on a finite interval $[\tau_-(x, v), \tau_+(x, v)]$, with

$$\gamma_{(x,v)}(\tau_-(x, v)) \in \partial M, \quad \gamma_{(x,v)}(\tau_+(x, v)) \in \partial M.$$

This defines two functions $\tau_\pm : TM_0 \to \mathbb{R}$ which plainly satisfy

$$\tau_-(x, v) \leq 0, \quad \tau_+(x, v) \geq 0,$$

$$\tau_+(x, v) = -\tau_-(x, -v),$$

$$\tau_-|_{\partial_+(SM)} = \tau_+|_{\partial_-(SM)} = 0,$$

$$\tau_-(\phi_t(x, v)) = \tau_-(x, v) - t, \quad \tau_+(\phi_t(x, v)) = \tau_+(x, v) + t \tag{2.3.1}$$

and for $t > 0$,

$$\tau_\pm(x, tv) = \frac{1}{t}\tau_\pm(x, v).$$

Using the implicit function theorem, we see that the functions $\tau_\pm$ are smooth near points (x, v) such that $\gamma_{(x,v)}(t)$ intersects ∂M transversely for $t = \tau_\pm(x, v)$. Since ∂M is strictly convex, this holds everywhere apart from $TM_0 \cap T(\partial M)$, and thus we conclude that the functions $\tau_\pm$ are smooth away from $TM_0 \cap T(\partial M)$. One can however show (see [**Sha94**, Lemma 4.1.1]) that the functions

$$\tau_+|_{\partial_+(SM)} : \partial_+(SM) \to \mathbb{R}, \quad \tau_-|_{\partial_-(SM)} : \partial_-(SM) \to \mathbb{R}$$

are smooth.

DEFINITION 2.11. The *scattering relation* is the map defined by

$$\vartheta : \partial_+(SM) \to \partial_-(SM);$$

$$\vartheta(x, v) := \phi_{\tau_+(x,v)}(x, v).$$

Suppose g and g' are two simple metrics on $(M, \partial M)$ such that $d = d'$ on $\partial M \times \partial M$ (where $d = d_g$ and $d' = d_{g'}$ etc.). We will show in Proposition 2.13 below that, modifying g' by a diffeomorphism which is the identity on the boundary if necessary, we have

$$\partial_\pm(SM) = \partial_\pm(S'M)$$

and thus the following definition makes sense:

DEFINITION 2.12. Let $(M, \partial M)$ be a compact manifold and g, g' two simple metrics on M. We say g and g' are *lens equivalent* if $\tau_+|_{\partial_+(SM)} = \tau'_+|_{\partial_+(S'M)}$ and $\vartheta = \vartheta'$. The *lens rigidity problem* asks whether lens rigidity determines the metric (up to a diffeomorphism which is the identity on the boundary).

PROPOSITION 2.13. *Let g and g' be two simple metrics on the compact manifold with boundary M such that $d = d'$ on $\partial M \times \partial M$. Then there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = Id$ and such that if $g'' := \psi^* g'$ then $g = g''$ on $T_{\partial M} M \times T_{\partial M} M$.*

PROOF. Let $(x, v) \in T(\partial M)$ and take a curve $\gamma : (-\epsilon, \epsilon) \to \partial M$ adapted to (x, v). Since γ takes values in ∂M, for all $s \in (-\epsilon, \epsilon)$ we have

$$d(x, \gamma(s)) = d'(x, \gamma(s)).$$

It follows that

$$|v| = \lim_{s \downarrow 0} \frac{d(x, \gamma(s))}{s} = \lim_{s \downarrow 0} \frac{d'(x, \gamma(s))}{s} = |v|'.$$

Thus by the polarization identity we see that already we have $g = g'$ on $T(\partial M) \times T(\partial M)$. This is not good enough though; in general we will need to modify g' before we obtain the stronger statement of the proposition.

Let $\nu(x)$ denote the unit inward pointing normal with respect to g, and define the *boundary exponential map*

$$\exp_{\partial M} : \partial M \times \{t \geq 0\} \to M, \qquad (x, t) \mapsto \exp_x(t\nu(x)),$$

which maps a neighborhood of $\partial M \times \{0\}$ diffeomorphically onto a neighborhood of ∂M. Now define

$$\psi := \exp'_{\partial M} \circ (\exp_{\partial M})^{-1} \tag{2.3.2}$$

(where the superscripts denote which metric they belong to). Then on some collar neighborhood U of ∂M, ψ is a diffeomorphism. It can be shown (although this requires a bit of effort) that it is possible to extend ψ smoothly across all of M. Assume this is done. Then we claim $\psi : M \to M$ satisfies the requirements of the proposition. Indeed, $\psi|_{\partial M} = \mathrm{Id}$, and moreover given $x \in \partial M$, if γ is the unique g-geodesic adapted to $(x, \nu(x))$ and similarly γ' is the unique g'-geodesic adapted to $(x, \nu'(x))$ then

$$\psi(\gamma(t)) = \gamma'(t).$$

Hence by differentiating we have

$$d_x\psi(\nu(x)) = \nu'(x).$$

Now define $g'' := \psi^* g'$. Then observe if $x \in \partial M$ and $v \in T_x\partial M$ then

$$\begin{aligned} g''(v, \nu(x)) &= g'(d_x\psi(v), d_x\psi(\nu(x))) \\ &= g'(v, \nu'(x)) \\ &= 0, \end{aligned}$$

since $d_x\psi|_{T_x\partial M} = \mathrm{Id}$, and thus g'' has unit inward normal vector field equal to ν. Next, for $x \in \partial M$ we have the decomposition

$$T_xM = T_x\partial M \oplus \mathbb{R}\nu(x),$$

since $T_x\partial M$ is a codimension 1 vector subspace of T_xM and $0 \neq \nu(x) \notin T_x\partial M$. Finally, since g' (and hence g'', since $d_x\psi$ is the identity on $T_x\partial M$) and g agree on $T\partial M \times T\partial M$, it follows that

$$g = g'' \quad \text{on} \quad T_{\partial M}M \times T_{\partial M}M,$$

as we wanted to show. □

The following result shows that lens rigidity is weaker than boundary rigidity.

PROPOSITION 2.14. *Let g and g' be two simple metrics on the compact manifold with boundary M, such that $d = d'$ on $\partial M \times \partial M$. Then after modifying g' by a diffeomorphism if necessary, the scattering relations coincide: $\vartheta \equiv \vartheta'$.*

PROOF. Let $(x, v) \in \partial_+(SM) = \partial_+(S'M)$. Suppose $\vartheta(x, v) = (y, w) \in \partial_-(SM)$. Let $\gamma_{x,y}$ denote the unique g-geodesic that starts at x and ends at y. Then we necessarily have $\dot\gamma_{x,y}(0) = v$ and $\dot\gamma_{x,y}(\tau_+(x, v)) = w$. Similarly let $\gamma'_{x,y}$ denote the unique g'-geodesic that starts at x and ends at y. Let

$$v' := \dot\gamma'_{x.y}(0), \quad w' := \dot\gamma'_{x,y}(\tau'_+(x, v')).$$

To complete the proof we will show that

$$v = v', \quad w = w'.$$

Indeed, we will then have $\vartheta'(x, v) = (y, w) = \vartheta(x, v)$. We begin by showing that $w = w'$.

Define the distance function $r : M \to \mathbb{R}$ by $r(p) = d(x, p)$ and similarly $r'(p) = d'(x, p)$. Then we claim that $\nabla r(y) = w$. Indeed, the Gauss Lemma shows that w is g-orthogonal with respect to the g-geodesic spheres $r = \text{const}$. But $\nabla r(y)$ has the same property, and thus w and $\nabla r(y)$ are parallel. Since both are unit vectors with outward pointing normal component, we must have $\nabla r(y) = w$ as claimed. Similarly $\nabla r'(y) = w'$.

Now let $h = r|_{\partial M}$. Then $\nabla h(y)$ is the orthogonal projection of the vector $\nabla r(y)$ in the 'hemisphere' $\partial_-(S_y M)$ onto the 'equator' $T_y \partial M$; in particular $\nabla h(y)$ uniquely determines $\nabla r(y)$. But by assumption, $h = h'$, and thus $\nabla h(y) = \nabla h'(y)$, and hence

$$w = \nabla r(y) = \nabla r'(y) = w'.$$

Now we show that $v = v'$. For this we simply repeat the same argument as above, only starting at y and running the two geodesics back towards x, using the fact that the unique geodesic (in either metric) from y to x is just the geodesic from x to y traversed backwards. This completes the proof. □

In fact, for simple metrics the boundary rigidity problem and the lens rigidity problem are equivalent. Indeed, it follows readily from the definitions that if $\tau_+ = \tau'_+$ on $\partial_+(SM) = \partial_+(S'M)$ and $\pi \circ \vartheta = \pi \circ \vartheta'$, then $d = d'$ on $\partial M \times \partial M$.

2.4. Equivalence of C^∞ jets

The following result is the main one of this chapter, and was proved by Lassas, Sharafutdinov and Uhlmann in [**LSU03**, Theorem 2.1]. The history of the result goes back earlier though; Michel proved the 2-dimensional case in [**Mic81**] and then later a C^2 jet version [**Mic94**] (as opposed to the C^∞ jet statement below).

THEOREM 2.15. *(Lassas, Sharafutdinov, Uhlmann* [**LSU03**]*)*
Let g and g' be two simple metrics on M^n such that $d = d'$ on $\partial M \times \partial M$. Then after modifying g' by a diffeomorphism which is the identity on the boundary if necessary, g and g' have the same C^∞-jet on ∂M.

PROOF. By Proposition 2.8 we may assume $g = g'$ on $T_{\partial M} M \times T_{\partial M} M$. Set $f := g - g'$. First observe that if I denotes the X-ray transform then

$$I[f](\gamma) \leq 0 \tag{2.4.1}$$

for all geodesics γ on (M, g) connecting boundary points. Indeed, if γ is a g-geodesic, we may assume γ is defined on $[0, 1]$ and compute

$$\begin{aligned} I[f](\gamma) &= \int_0^1 f_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt \\ &= \int_0^1 g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt - \int_0^1 g'_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt \\ &\leq d(x, y)^2 - d'(x, y)^2 \\ &= 0. \end{aligned}$$

Fix a point $p \in \partial M$ and take *boundary normal coordinates* $(u^1, \dots, u^{n-1}, z)$ on a neighborhood U of p. By definition these are coordinates such that $z \geq 0$ on U and $\partial M \cap U = \{z = 0\}$, and that the length element ds^2 of the metric g is given by

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta + dz^2, \qquad \alpha, \beta \in \{1, \dots, n-1\},$$

(the Gauss Lemma shows that such coordinates always exist). Now the coordinate lines $u =$ const are geodesics of the metric g orthogonal to the boundary; since $g = g'$ on $T_{\partial M} M \times T_{\partial M} M$ it follows that the *same* coordinates are also boundary normal coordinates for g'; in particular

$$ds'^2 = g'_{\alpha\beta} du^\alpha du^\beta + dz^2, \qquad \alpha, \beta \in \{1, \dots, n-1\}.$$

Since p was arbitrary, to complete the proof it suffices to show that for all $x \in U \cap \partial M$, $k \in \mathbb{N} \cup \{0\}$ and $1 \leq \alpha, \beta \leq n-1$ we have

$$\frac{\partial f_{\alpha\beta}}{\partial z^k}(x) = 0, \tag{2.4.2}$$

where $f_{\alpha\beta} = g_{\alpha\beta} - g'_{\alpha\beta}$. The case $k = 0$ is precisely the assertion that $g = g'$ on $T_{\partial M} M \times T_{\partial M} M$, and so this gives the base step for an inductive proof. Suppose that (2.4.2) holds for $0 \leq k < \ell$ but fails for ℓ.

This implies the existence of $x_0 \in \partial M \cap U$ and $v_0 \in S_x \partial M$ such that

$$\frac{\partial^\ell f_{\alpha\beta}}{\partial z^\ell}(x_0) v_0^\alpha v_0^\beta \neq 0.$$

Without loss of generality we may assume

$$\frac{\partial^\ell f_{\alpha\beta}}{\partial z^\ell}(x_0) v_0^\alpha v_0^\beta > 0.$$

By continuity of f, there exists a neighborhood $\mathbb{O} \subseteq SM$ of (x_0, v_0) such that for all $(x, v) \in \mathbb{O}$,

$$\frac{\partial^\ell f_{\alpha\beta}}{\partial z^\ell}(x) v^\alpha v^\beta > 0. \tag{2.4.3}$$

Since (2.4.3) is a homogeneous polynomial of degree 2, we may assume that if

$$C\mathbb{O} := \left\{ (x, v) \in TM \ : \ v \neq 0, \left(x, \frac{v}{|v|} \right) \in \mathbb{O} \right\}$$

then (2.4.3) holds for all $(x, v) \in C\mathbb{O}$.

Now we develop $f_{\alpha\beta}$ in a Taylor series; using the inductive hypotheses we may write

$$f_{\alpha\beta}(u, z) = \frac{1}{\ell!} \frac{\partial^\ell f_{\alpha\beta}}{\partial z^\ell}(u, 0) z^\ell + o(|z|^\ell),$$

and hence shrinking $\mathbb{O}$ if necessary we may assume that for all $(x, v) \in C\mathbb{O}$ we actually have

$$f_{\alpha\beta}(x) v^\alpha v^\beta > 0. \tag{2.4.4}$$

Now let $\delta : (-\epsilon, \epsilon) \to M$ be a curve adapted to (x_0, v_0), and let $\gamma_\tau : [0, 1] \to M$ be the shortest geodesic of g joining x_0 and $\delta(\tau)$. Then

$$\left(\gamma_\tau(t), \frac{\dot\gamma_\tau(t)}{|\dot\gamma_\tau(t)|} \right) \to (x_0, v_0) \text{ uniformly in } t \in [0, 1] \text{ as } \tau \to 0.$$

Thus for sufficiently small $\tau > 0$, we have $(\gamma_\tau(t), \dot\gamma_\tau(t)) \in C\mathbb{O}$ for all $t \in [0, 1]$, and thus for τ sufficiently small we have

$$I[f](\gamma_\tau) > 0,$$

contradicting (2.4.1). This completes the proof. □

2.5. Solving the problem in the real analytic case

As a corollary we present the following result, also due to Lassas, Sharafutdinov and Uhlmann ([**LSU03**, Theorem 2.2]) that solves the boundary rigidity conjecture in the real analytic case.

THEOREM 2.16. *Let M be a compact real analytic manifold with real analytic boundary ∂M. Let g and g' be two simple real analytic Riemannian metrics on M such that $d = d'$ on $\partial M \times \partial M$. Then there exists a real analytic diffeomorphism $\psi : M \to M$ with $\psi|_{\partial M} = Id$ and $g' = \psi^* g$.*

PROOF. As in (2.3.2) there exists a collar neighborhood U of ∂M such that $\psi := \exp'_{\partial M} \circ (\exp_{\partial M})^{-1}$ is an injective local diffeomorphism, which is obviously real analytic for real analytic metrics. As shown in the proof of Theorem 2.15, g and $\psi^* g'$ have the same C^∞-jet on ∂M. Thus by analyticity, g and $\psi^* g'$ coincide in some connected neighborhood of ∂M, and ψ is a real analytic isometry of the real analytic manifold (U, g) onto the real analytic manifold (U, g') which is the identity when restricted to ∂M.

To complete the proof we need to show that ψ extends to a real analytic isometry $\widehat{\psi}$ of (M, g) onto (M, g'). The argument is somewhat intricate and beyond the scope of these notes; we refer the reader to [**LU89**, Theorem C(a)]. □

2.6. Gluing

We conclude this chapter by discussing an application due to Croke [**Cro04**] of the previous result.

EXAMPLE 2.17. Let (N, g) be a closed manifold, and suppose $(M, \partial M)$ is an embedded submanifold of N. Then the metric g restricts to M to give a metric (still denoted by) g on M. Suppose we have another metric g' on M, and suppose both g and g' are simple and such that $d = d'$ on $\partial M \times \partial M$. Then Theorem 2.15 tells us that, after modifying g' by a diffeomorphism if necessary, we may *glue* $(M, \partial M, g')$ into N to get a new Riemannian manifold (N, g'). More precisely, we can define a new Riemannian metric g' on N by

$$g' = \begin{cases} g' & M \\ g & N \backslash M. \end{cases}$$

DEFINITION 2.18. Suppose N and N' are smooth manifolds with flows ϕ_t and f_t respectively. A diffeomorphism of class C^r, $\psi : N \to N'$ is called a *C^r-time preserving conjugacy* if ψ intertwines the flows, that is,

$$f_t \circ \psi = \psi \circ \phi_t.$$

If such a map ψ exists we say that ϕ_t and f_t are C^r-conjugate. A related concept is a non time preserving conjugacy, called an *orbit equivalence*, which is a diffeomorphism $\psi : N \to N'$ that takes the orbits of ϕ_t into the orbits of f_t. Occasionally we will want to consider merely continuous conjugacies.

LEMMA 2.19. *In the situation described in Example 2.17, the geodesic flows of* (N, g) *and* (N, g') *are* C^∞*-conjugate.*

PROOF. Let SN and $S'N$ denote the unit sphere bundles of N with respect to g and g' respectively, and similarly for SM and $S'M$. Let ϕ_t and f_t denote the corresponding geodesic flows on N. Suppose $(x, v) \in SM$. Define $\psi : SM \to S'M$ by

$$\psi(x, v) = f_{-\tau_-(x,v)} \circ \phi_{\tau_-(x,v)}(x, v),$$

where $\tau_-(x, v)$ is the function defined in Section 2.10, with respect to g.

We claim $\psi|_{\partial(SM)} = \mathrm{Id}$. Indeed, since $\tau_-|_{\partial_+(SM)} = 0$, we certainly have $\psi|_{\partial_+(SM)} = \mathrm{Id}$, and since the scattering relations of $(M, \partial M, g)$ and $(M, \partial M, g')$ coincide by Proposition 2.14 we also have $f_{-\tau_-(x,v)} \circ \phi_{\tau_-(x,v)} = \mathrm{Id}$ for $(x, v) \in \partial_-(SM)$. Let us extend ψ to a homeomorphism $SN \to S'N$ by letting it be the identity outside of SM. To see that ψ is smooth we proceed as follows. Let U be any open neighborhood of M in N such that any g-geodesic $\gamma_{(x,v)}$ with $(x, v) \in SU$ intersects $\partial\bar{U}$ transversally. Then we have a length $\tau_-^{\bar{U}}$ which is smooth on SU. Since the map $f_{-\tau_-^{\bar{U}}(x,v)} \circ \phi_{\tau_-^{\bar{U}}(x,v)}$ defined on SU is smooth and coincides with ψ on SU, ψ is also smooth.

Now it remains only to check that ψ is actually a time preserving conjugacy, and for this it is enough to check on SM. Given $(x, v) \in SM$ and $t \in \mathbb{R}$, observe firstly that by (2.3.1),

$$\tau_-(\phi_t(x, v)) = \tau_-(x, v) - t,$$

and thus

$$\begin{aligned} \psi(\phi_t(x, v)) &= f_{-\tau_-(\phi_t(x,v))} \circ \phi_{\tau_-(\phi_t(x,v))}(\phi_t(x, v)), \\ &= f_{t-\tau_-(x,v)} \circ \phi_{\tau_-(x,v)-t} \circ \phi_t(x, v) \\ &= f_t \circ \left\{ f_{-\tau_-(x,v)} \circ \phi_{\tau_-(x,v)}(x, v) \right\} \\ &= f_t(\psi(x, v)). \end{aligned}$$

□

This lemma will give us a way of showing that certain manifolds with boundary are boundary rigid by using rigidity results for geodesic flows on closed manifolds. We will return to this later on, but we finish this chapter by showing that the existence of a C^1-conjugacy implies equality of volumes. We first prove the following technical lemma, due originally to Croke and Kleiner ([**CK94**, Lemma 2.1]).

LEMMA 2.20. *Let P be a compact $(2n-1)$-manifold either with or without boundary, and θ_0, θ_1 1-forms on P and X a vector field on P. Suppose that*

$$\theta_0(X) = \theta_1(X) = 1,$$

$$i_X d\theta_0 = i_X d\theta_1 = 0,$$

and if $\partial P \neq \emptyset$ then $\theta_0|_{\partial P} = \theta_1|_{\partial P}$ (if $\partial P = \emptyset$ then this condition is vacuous).

Then

$$\int_P \theta_0 \wedge (d\theta_0)^{n-1} = \int_P \theta_1 \wedge (d\theta_1)^{n-1}. \tag{2.6.1}$$

PROOF. Let $\theta_t := t\theta_1 + (1-t)\theta_0$. Then clearly $\theta_t(X) = 1$ and $i_X d\theta_t = 0$ for all t, and letting $\dot{\theta}_t$ denote $\frac{\partial}{\partial t}\theta_t$ we have $\dot{\theta}_t = \theta_1 - \theta_0$; in particular $\dot{\theta}_t|_{\partial P} = 0$ if $\partial P \neq \emptyset$. We will show that

$$\frac{d}{dt}\int_P \theta_t \wedge (d\theta_t)^{n-1} = 0,$$

whence the result follows. Indeed,

$$\frac{d}{dt}\int_P \theta_t \wedge (d\theta_t)^{n-1} = \int_P \dot{\theta}_t \wedge (d\theta_t)^{n-1} + (n-1)\int_P \theta_t \wedge d\dot{\theta}_t \wedge (d\theta_t)^{n-2}. \tag{2.6.2}$$

But now since $i_X(\dot{\theta}_t \wedge (d\theta_t)^{n-1}) = 0$, as $\dot{\theta}_t \wedge (d\theta_t)^{n-1}$ is a top dimensional form this implies $\dot{\theta}_t \wedge (d\theta_t)^{n-1} = 0$. Next, we note that

$$\begin{aligned} d\left(\theta_t \wedge \dot{\theta}_t \wedge (d\theta_t)^{n-2}\right) &= \dot{\theta}_t \wedge (d\theta_t)^{n-1} - \theta_t \wedge d\dot{\theta}_t \wedge (d\theta_t)^{n-2} + \theta_t \wedge \dot{\theta}_t \wedge d(d\theta_t)^{n-2} \\ &= -\theta_t \wedge d\dot{\theta}_t \wedge (d\theta_t)^{n-2}. \end{aligned}$$

This shows that the first integral in (2.6.2) is zero, and the second is the integral of an exact form, and thus by Stokes' Theorem is equal to

$$-(n-1)\int_{\partial P} \theta_t \wedge \dot{\theta}_t \wedge (d\theta_t)^{n-2},$$

which is certainly zero if $\partial P = \emptyset$, and if $\partial P \neq \emptyset$ it is also zero since $\dot{\theta}_t|_{\partial P} = 0$. □

The following result is from [**CK94**, Proposition 1.2]

PROPOSITION 2.21. *Let N be a closed manifold and g, g' two metrics on N such that the geodesic flows of (N, g) and (N, g') are smoothly conjugate. Then Vol(N, g) = Vol(N, g').*

PROOF. Let $\psi : SN \to S'N$ denote the conjugacy, and let ϕ_t and f_t denote the corresponding geodesic flows, with infinitesimal generators X and X'. Let α and α' denote the respective contact forms. Now let $\theta_0 = \alpha$ and $\theta_1 = \psi^*\alpha'$.

Then

$$\begin{aligned} \theta_1(X) &= \psi^*\alpha'(X) \\ &= \alpha'(d\psi(X)) \\ &\overset{(*)}{=} \alpha'(X') \\ &= 1, \end{aligned}$$

where $(*)$ follows by differentiating $f_t \circ \psi = \psi \circ \phi_t$. Similarly, it is easy to check that $i_X d\theta_1 = 0$ and hence Lemma 2.20 implies

$$\begin{aligned} \int_{SN} \alpha \wedge (d\alpha)^{n-1} &= \int_{SN} \psi^*\alpha' \wedge (d(\psi^*\alpha'))^{n-1} \\ &= \int_{SN} \psi^*(\alpha' \wedge (d\alpha')^{n-1}) \\ &= \int_{S'N} \alpha' \wedge (d\alpha')^{n-1}. \end{aligned}$$

To complete the proof we quuote the following fact, which we shall need at several points in the course (here g_{round} denotes the canonical round metric on S^{n-1}):

$$\text{Vol}(N, g) = \frac{(-1)^{n-1}}{(n-1)!\text{Vol}(S^{n-1}, g_{\text{round}})}\int_{SN} \alpha \wedge (d\alpha)^{n-1}. \tag{2.6.3}$$

A reference for this formula is [**Cha06**, Theorem VII.1.3]. Here we are using the convention that if $\dim N = n$ then $(-1)^{n-1}\alpha \wedge (d\alpha)^{n-1}$ is a positively oriented volume form on SN, which is consistent with the conventions made in the rest of these notes. □

In fact, one only needs the conjugacy to be of class C^1 to obtain equality of volume; the proof above easily extends, see [**CK94**].

Finally, an immediate corollary of Lemma 2.20 and Proposition 2.21 is the following.

COROLLARY 2.22. *In the situation described in Example 2.17 we have* $Vol(N, g) = Vol(N, g')$.

EXERCISE 2.23. Using the same ideas as above show that two simple manifolds with the same boundary distance function must have the same volume.

CHAPTER 3

Anosov flows

This chapter surveys some general results about Anosov flows. In the first section we consider Anosov diffeomorphisms and give a few examples. We then move on to discussing the Anosov property in general, and conclude with a relevent theorem due to Ghys.

3.1. Anosov diffeomorphisms

To begin with we wish to give a few examples of the Anosov condition. We will describe the related concept of an Anosov diffeomorphism; essentially the discrete-time analogue of the above. Any Anosov diffeomorphism induces an Anosov flow on its suspension.

DEFINITION 3.1. Contrast this definition with the definition of an Anosov flow in Section 1.2. Given a smooth manifold M and $U \subseteq M$ open, and $\psi : U \to M$ a C^1 diffeomorphism onto its image, and $\Lambda \subseteq U$ a compact ψ-invariant set, we say that Λ is a *hyperbolic set* for ψ if there exists a metric (called a *Lyapunov metric*) on an open neighborhood V of Λ and $\lambda < 1 < \mu$ such that for any $x \in \Lambda$ and any $n \in \mathbb{Z}$, the differential

$$d_{\psi^n(x)}\psi : T_{\psi^n x}M \to T_{\psi^{n+1}x}M$$

induces a splitting

$$T_{\psi^n(x)}M = E^s(\psi^n x) \oplus E^u(\psi^n x),$$

such that

$$d_{\psi^n x}\psi(E^s(\psi^n x)) \subseteq E^s(\psi^{n+1}x),$$
$$d_{\psi^n x}\psi(E^u(\psi^n x)) \subseteq E^u(\psi^{n+1}x)),$$

and such that

$$\left\| d_{\psi^n x}\psi|_{E^s(\psi^n x)} \right\| \leq \mu^{-n},$$
$$\left\| d_{\psi^n x}\psi|_{E^u(\psi^n x)} \right\| \leq \lambda^{n}.$$

If $\psi : M \to M$ is a smooth map of the compact manifold M we say that ψ is an *Anosov diffeomorphism* if M is itself a hyperbolic set for ψ.

EXAMPLE 3.2. Let $A : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z} \to \mathbb{T}^2$ be given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then A is an Anosov diffeomorphism. Indeed, taking the standard Euclidean metric on $\mathbb{T}^2$, since the eigenvalues of A are

$$\lambda = \frac{3-\sqrt{5}}{2}, \qquad \mu = \frac{3+\sqrt{5}}{2},$$

and we obtain a hyperbolic splitting of $T_{A^n x}\mathbb{T}^2$ given by the eigenspaces of A.

DEFINITION 3.3. Let $\psi : M \to M$ be a diffeomorphism. We can construct a *suspension flow* on the *suspension manifold* M_ψ as follows.

Let M_ψ denote the quotient space of $M \times \mathbb{R}$ obtained by identifying

$$(x,t) \sim (\psi x, t+1).$$

Define $\phi_t : M \times \mathbb{R} \to M \times \mathbb{R}$ by $\phi_t(x,s) = (x, s+t)$. Then ϕ_t induces a flow ϕ_t^ψ on the quotient manifold M_ψ.

EXERCISE 3.4. Show that if ψ is an Anosov diffeomorphism on N, then ϕ_t^ψ is an Anosov flow on M.

Hence the 3-manifold M_A where A is as in the previous example carries an Anosov flow ϕ_t^A.

EXAMPLE 3.5. Let Γ be a discrete subgroup of $PSL(2,\mathbb{R})$, such that $N := \Gamma \setminus PSL(2,\mathbb{R})$ is compact (Γ is called a *cocompact lattice* or a *uniform lattice*).

The Lie algebra of $PSL(2,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{R})$, and we may take as a basis of $\mathfrak{sl}(2,\mathbb{R})$ the set $\{X, H, V\}$ where

$$X := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad H := \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{3.1.1}$$

Note that we have the *structural equations*

$$\begin{aligned} [V, X] &= H \\ [H, V] &= X \\ [X, H] &= -V \end{aligned} \tag{3.1.2}$$

Given any $A \in \mathfrak{sl}(2,\mathbb{R})$, we may write $A = xX + yH + zV$. We define a map sending $A = xX + yH + zV$ to $a = x^2 + y^2 - z^2$, and then we consider the cone in $\mathbb{R}^3$ determined by $\{x^2 + y^2 - z^2 = 0\}$. Given $A \in \mathfrak{sl}(2,\mathbb{R})$, we say that A is *hyperbolic* if a lies outside the cone, that is, $a > 0$, we say A is *elliptic* if a lies inside the cone, that is, $a < 0$ and we say A is *parabolic* if a lies on the cone, that is, $a = 0$. Thus X and H are hyperbolic, V is elliptic and $H + V$ and $H - V$ are parabolic.

We can define a map $\phi_t : N \to N$ by $\phi_t(\Gamma g) = \Gamma(g \cdot \exp(tX))$, where $g \in PSL(2,\mathbb{R})$ and Γg denotes the orbit of g under Γ.

LEMMA 3.6. *The flow ϕ_t is Anosov.*

PROOF. Since $[X, H + V] = -(H + V)$ and $[X, H - V] = H - V$ and $\{X, H + V, H - V\}$ is a basis of $\mathfrak{sl}(2,\mathbb{R})$, we obtain a splitting of $T_{\Gamma g}N$ as

$$T_{\Gamma(g)}N = \mathbb{R}X(\Gamma g) \oplus E^s(\Gamma g) \oplus E^u(\Gamma g),$$

where

$$E^s(\Gamma g) = \operatorname{span}\left\{ \frac{d}{dt}\Big|_{t=0} a_t(\Gamma g) \right\},$$

$$E^u(\Gamma g) = \operatorname{span}\left\{ \frac{d}{dt}\Big|_{t=0} b_t(\Gamma g) \right\},$$

with

$$a_t : \Gamma g \mapsto \Gamma(g \cdot \exp(t(H - V)))$$

and

$$b_t : \Gamma g \mapsto \Gamma(g \cdot \exp(t(H + V))).$$

Since for $t \geq 0$,

$$\phi_t \circ a_s = a_{se^{-t}} \circ \phi_t,$$

and

$$\phi_t \circ b_s = b_{se^t} \circ \phi_t,$$

we see that E^s and E^u are $d\phi_t$-invariant and for all $t \in \mathbb{R}$,

$$\|d\phi_t|_{E^s}\| = e^{-t},$$

$$\|d\phi_t|_{E^u}\| = e^t,$$

where the norms is defined by declaring that X, H, V is an orthonormal basis.

□

REMARK 3.7. We can use this to prove a special case of Theorem 1.6. Let M be a complete Riemannian surface of constant curvature -1 with finite area. Then the universal cover of M is $\mathbb{H}^2$, the hyperbolic plane. The fundamental group of M acts on $\mathbb{H}^2$ by a discrete group $\Gamma \subseteq PSL(2,\mathbb{R})$ of hyperbolic isometries, and $M \cong \Gamma \setminus \mathbb{H}^2$. The unit sphere bundle of $\mathbb{H}^2$ is isomorphic to $PSL(2,\mathbb{R})$, and hence the unit sphere bundle SM is $\Gamma \setminus PSL(2,\mathbb{R})$.

Now if $\phi_t : SM \to SM$ denotes the geodesic flow of M, then it can be shown that

$$\phi_t(\Gamma g) = \Gamma(g \cdot \exp(tX)).$$

Hence the previous lemma shows that ϕ_t is indeed Anosov.

3.2. General theory

In this subsection we shall collect together various facts (some without proof) about Anosov flows that we will need in the sequel.

DEFINITION 3.8. Let N be a closed manifold and $\phi_t : N \to N$ an Anosov flow. Define the *stable manifold of ϕ_t at x* and the *unstable manifold of ϕ_t at x* to be the sets

$$W^s(x) = \{y \in N \,:\, d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to \infty\},$$

$$W^u(x) = \{y \in N \,:\, d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to -\infty\}.$$

In fact, in both cases we could instead require the convergence to be exponentially fast, that is, $d(\phi_t x, \phi_t y) \leq Ce^{-\lambda t}$ say, for $y \in W^s(x)$.

Let $\phi_t : N \to N$ be an Anosov flow on a closed manifold N, and let $x \in N$ and $U \subseteq N$ a neighborhood of x. We define the *local stable manifold of ϕ_t at x with respect to U* and the *local unstable manifold of ϕ_t at x with respect to U* to be

$$W^s_{\text{loc}}(x, U) := \{y \in N \,:\, \phi_t y \in U \text{ for all } t \geq 0,\ d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to \infty\},$$
$$W^u_{\text{loc}}(x, U) := \{y \in N \,:\, \phi_t y \in U \text{ for all } t \leq 0,\ d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to -\infty\}.$$

The next theorem is an important result called the *Local stable manifold theorem*, for which we refer the reader to [**KH95**, Theorem 17.43] for a proof. In particular this next theorem justifies the use of the word 'manifold' above.

THEOREM 3.9. *Let $\phi_t : N \to N$ be an Anosov flow on a closed manifold N. Then there exists $\varepsilon > 0$ such that for each $x \in N$ the local (un)stable manifolds*

$$W^s_{\text{loc}}(x) := W^s_{\text{loc}}(x, B(x,\varepsilon)), \quad W^u_{\text{loc}}(x) := W^u_{\text{loc}}(x, B(x,\varepsilon))$$

are embedded discs such that

$$T_x W^s_{\text{loc}}(x) = E^s(x), \quad T_x W^u_{\text{loc}}(x) = E^u(x)$$

for all $x \in N$, and such that

$$\phi_t(W^s_{\text{loc}}(x)) \subseteq W^s_{\text{loc}}(\phi_t x), \quad \phi_t(W^u_{\text{loc}}(x)) \subseteq W^u_{\text{loc}}(\phi_t x)$$

for all $t > 0$.

In the sequel (principally Chapter 6) when we refer to $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(x)$, we will always mean this as shorthand for $W^s_{\text{loc}}(x, B(x,\varepsilon))$ and $W^u_{\text{loc}}(x, B(x,\varepsilon))$, where $\varepsilon > 0$ is chosen to satisfy the conditions of Theorem 3.9.

It is easy to see that for any neighborhood $U(t)$ of $\phi_t x$,

$$W^s(x) = \bigcup_{t>0} \phi_{-t}(W^s_{\text{loc}}(\phi_t x, U(t))), \tag{3.2.1}$$

$$W^u(x) = \bigcup_{t>0} \phi_t(W^u_{\text{loc}}(\phi_{-t} x, U(t))), \tag{3.2.2}$$

and thus in particular the right-hand sides of (3.2.1) and (3.2.2) are independent of the choice of neighborhood U. This gives us:

COROLLARY 3.10. *Let $\phi_t : N \to N$ by an Anosov flow on a closed manifold N. Then for each $x \in N$ the (un)stable manifolds $W^s(x), W^u(x)$ are injectively immersed submanifolds such that*

$$T_x W^s(x) = E^s(x), \quad T_x W^u(x) = E^u(x).$$

In general the stable and unstable bundles E^s and E^u of TN are only Hölder continuous for an Anosov flow $\phi_t : N \to N$ (see Chapter 11). However we have just shown that they are always integrable.

DEFINITION 3.11. Let N be a closed manifold and ϕ_t a flow on N. We define the *non-wandering set* Ω of ϕ_t to be the set

$$\Omega := \{x \in N \,:\, \text{for any open } U \ni x,\ \text{there exists } T \geq 1/2 \text{ such that } \phi_T(U) \cap U \neq \emptyset\}.$$

Clearly Ω is ϕ_t-invariant.

We will not try and prove the following theorem; for a proof see [**KH95**, Section 6.4, Section 18.3].

THEOREM 3.12. *Let $\phi_t : N \to N$ be an Anosov flow on a closed manifold N. Then $\Omega = N$ if and only if ϕ_t is transitive. Moreover, the periodic orbits of ϕ_t are dense in Ω.*

In general, transitivity trivially implies that $\Omega = N$; the converse however requires the Anosov condition and is much harder to prove.

LEMMA 3.13. *Let $\phi_t : N \to N$ be a flow on a closed manifold N. Suppose ϕ_t preserves a probability measure μ on N which is positive on open sets. Then $\Omega = N$.*

PROOF. Suppose to the contrary. Then there exists $x \in N$ and an open set U containing x such that the sets $\{U, \phi_1(U), \phi_2(U), \dots\}$ are all pairwise disjoint. Hence if $\mu(U) = \mu(\phi_n(U)) = c > 0$ then

$$\mu\left(\coprod_{n=0}^{\infty} \phi_n(U)\right) = \sum_{n=0}^{\infty} c = \infty,$$

which is absurd, since μ is a probability measure. □

COROLLARY 3.14. *If M has finite volume then the set $\Omega \subseteq SM$ of non-wandering points with respect to the geodesic flow ϕ_t is all of SM.*

PROOF. Recall that the contact form α on SM gives a volume form $\alpha \wedge (d\alpha)^{n-1}$ which if M has finite volume gives SM finite volume by (2.6.3). Thus α defines a finite measure μ called the *Liouville measure* on SM, which clearly is positive on open sets. The result follows from the previous lemma, since the geodesic flow ϕ_t preserves the contact form α and hence the measure μ (this is proved explicitly for surfaces in Lemma 9.2). □

REMARK 3.15. In the proof above we defined a finite measure μ on SM called the *Liouville measure*, given explicitly by

$$\int_{SM} f d\mu = \int_{SM} (-1)^{n-1} f\alpha \wedge (d\alpha)^{n-1} \quad f \in C^0(SM, \mathbb{R}).$$

Using equation (2.6.3), we see that

$$\mu(SM) = (n-1)! \mathrm{Vol}(M, g) \mathrm{Vol}(S^{n-1}, g_{\mathrm{round}}).$$

It is often convenient to normalize μ to form the *normalized Liouville measure* $\mu' := \mu/\mu(SM)$. The normalized Liouville measure μ' is therefore a probability measure. Throughout the text however we will not distinguish between μ and μ', referring to them both as 'the' Liouville measure, and denoted both by μ. It should be clear from the context whether we are normalizing or not.

Since $\phi_t^* \alpha = \alpha$, the Liouville measure is invariant under the geodesic flow, that is,

$$\mu(B) = \mu(\phi_t(B))$$

for any Borel measurable set $B \subseteq SM$. In fact, if ϕ_t is Anosov then μ is the unique smooth finite measure on SM that is ϕ_t-invariant. In Chapter 10 we shall see further evidence of the importance of μ; see for instance Corollary **??** and Theorem 10.55.

3.3. A classification result

DEFINITION 3.16. Let N be a closed 3-manifold. We say that N is a *circle bundle* if there exists a fibration $\pi : N \to M$ where M is a closed surface, such that $\pi^{-1}(x) \cong S^1$ for all $x \in M$. One obvious example is when $N = SM$.

We will use the following classification theorem several times throughout these notes. This was proved by Ghys in 1984 (see [**Ghy84**]) and similar ideas appear also in [**Gro00**].

THEOREM 3.17. *Let N be a closed 3-manifold that is a circle bundle. Suppose $\phi_t : N \to N$ is an Anosov flow. Then there exists a closed surface M of genus $\mathrm{g} \geq 2$ such that N is a finite cover of SM, and ϕ_t is continuously orbit equivalent to the lift to N of the geodesic flow on SM corresponding to a metric g_0 of constant negative curvature -1.*

In order to prove Theorem 3.17 we will need to introduce several new concepts. To begin with, let (M, g_0) denote an n-dimensional Riemannian manifold, where g_0 is a metric of constant negative curvature on M. Let $\tilde{M}$ denote the universal cover of M.

DEFINITION 3.18. The *ideal boundary* $\tilde{M}(\infty)$ of $(\tilde{M}, \tilde{g}_0)$ is given by $\tilde{M}(\infty) := \mathscr{G}(\tilde{M}, \tilde{g}_0)/\sim$, where $\mathscr{G}(\tilde{M}, \tilde{g}_0)$ denotes the set of $\tilde{g}_0$-geodesics $\gamma : \mathbb{R} \to \tilde{M}$ of $\tilde{M}$, and $\gamma_1 \sim \gamma_2$ if and only if

$$\lim_{t\to\infty} d_{\tilde{g}_0}(\gamma_1(t), \gamma_2(t)) \quad \text{remains bounded.}$$

Given $x \in \tilde{M}$ and $v \in T_x\tilde{M}$, let $\gamma_{(x,v)} : \mathbb{R} \to \tilde{M}$ denote the unique g_0-geodesic adapted to (x, v), and let $\gamma_{(x,v)}(\infty) \in \tilde{M}(\infty)$ denote the corresponding element of $\tilde{M}(\infty)$. If $\gamma_{(x,v)}^{-1} = \gamma_{(x,-v)}$ is the geodesic obtained by going along $\gamma_{(x,v)}$ backwards, let $\gamma_{(x,v)}(-\infty)$ denote the element of $\tilde{M}(\infty)$ corresponding to $\gamma_{(x,v)}^{-1}$.

Fix a point $x \in \tilde{M}$, and consider the map $s_x : S_x\tilde{M} \to \tilde{M}(\infty)$ sending $v \mapsto \gamma_v(\infty)$. Then s_x is a bijection, and we define a topology on $\tilde{M}(\infty)$ so that s_x becomes a homeomorphism; thus $S\tilde{M} \cong \tilde{M} \times \tilde{M}(\infty)$ and $\tilde{M}(\infty) \cong S^{n-1}$. This topology is independent of the choice of x, since $s_y \circ s_x^{-1} : S_x\tilde{M} \to S_y\tilde{M}$ is a homeomorphism.

If we regard $\pi_1(M)$ as a subgroup of $\mathrm{Isom}(\tilde{M}, \tilde{g}_0)$, then we obtain a natural action of $\pi_1(M)$ on $\tilde{M}(\infty)$ as follows. If $\xi \in \tilde{M}(\infty)$, choose $x \in \tilde{M}$ and $v \in T_x\tilde{M}$ such that $\xi = \gamma_{(x,v)}(\infty)$. Given $\varphi \in \pi_1(M)$, define $\varphi \cdot \xi$ to be the element of $\tilde{M}(\infty)$ defined by the curve $\varphi \circ \gamma_{(x,v)}$.

The next exercise relates the action of $\pi_1(M)$ on $\tilde{M}(\infty)$ with SM.

EXERCISE 3.19. Show that

$$SM \cong \tilde{M} \times \tilde{M}(\infty)/(x, \xi) \sim (\varphi(x), \varphi \cdot \xi).$$

REMARK 3.20. We will actually only be interested in the case where M is a closed surface of genus $\mathrm{g} \geq 2$. In this case, the universal cover $\tilde{M}$ is the Poincaré disk (B^2, g_{hyp}), where here g_{hyp} denotes the hyperbolic metric

$$g_{\mathrm{hyp}} = \frac{4g_{\mathrm{eucl}}}{(1 - r^2)^2},$$

where r denotes the Euclidean distance to the origin, and the ideal boundary $\tilde{M}(\infty)$ is precisely the boundary circle $S^1 = \partial B^2$.

The next thing we require is the concept of a quasi-geodesic.

DEFINITION 3.21. A curve $\gamma : [a, b] \to \tilde{M}$ is a $\tilde{g}_0$-*quasi-geodesic* of $(\tilde{M}, \tilde{g}_0)$ if there exist $P, Q \in \mathbb{R}^+$ such that

$$\frac{1}{P}|s - t| - Q \leq \mathrm{dist}_{\tilde{g}_0}(\gamma(s), \gamma(t)) \leq P|s - t| + Q$$

for all $s, t \in [a, b]$. If we need to be explicit about the constants P, Q, we call such a quasi-geodesic a (P, Q)-$\tilde{g}_0$-*quasi-geodesic*.

We can build the *quasi-ideal boundary* $\tilde{M}^*(\infty)$ in much the same way using quasi-geodesics. If $\gamma : \mathbb{R} \to \tilde{M}$ is a $\tilde{g}_0$-quasi-geodesic, we write $\gamma^*(\infty)$ to denote the corresponding element of $\tilde{M}^*(\infty)$. Note that any geodesic is automatically a quasi-geodesic, and thus we have a natural map $\tilde{M}(\infty) \hookrightarrow \tilde{M}^*(\infty)$ carrying an equivalence class of geodesics to the corresponding equivalence class of quasi-geodesics. The following theorem is essentially originally due to Morse [**Mor24**], and is often known as the *Morse lemma*. A proof may be found in [**Kni02**, Theorem 2.2].

THEOREM 3.22. *The inclusion* $\tilde{M}(\infty) \hookrightarrow \tilde{M}^*(\infty)$ *is a bijection.*

We will next need to recall the definition and some of the basic theory of *foliations*. A good reference on the subject is [**CN85**], to which we refer the reader for more information. Another nice reference is [**Cal07**], which concentrates on the case we are interested in here: 3-manifolds.

DEFINITION 3.23. Let N be a smooth closed n-dimensional manifold. Let B^k denote the open unit ball in $\mathbb{R}^k$. A partition $\mathscr{F}$ of N into connected k-dimensional C^1-submanifolds $\{F(x) : x \in N\}$ is called a k-dimensional *foliation* if for every $x \in N$ there exists a neighborhood $U(x)$ of x and a diffeomorphism $\varphi : U(x) \to B^k \times B^{n-k}$ such that $\varphi(x) = (0, 0)$ and:

(1) For each $p \in B^{n-k}$, the set $\varphi(B^k \times \{p\})$ is precisely the connected component of $F(\varphi(0, p)) \cap U(x)$ containing $\varphi(0, p)$.
(2) For each $p \in B^{n-k}$, the map $\varphi(\cdot, p) : B^k \to F(\varphi(0, p)) \cap U(x)$ is a C^1-diffeomorphism which depends continuously on $p \in B^{n-k}$ in the C^1-topology.

Strictly speaking, we just defined a k-dimensional C^1-foliation. The sets $F(x)$ are called the *leaves* of $\mathcal{F}$.

The main examples we have in mind are the (un)stable foliations defined by an Anosov flow. More precisely, suppose $\phi_t : N \to N$ is an Anosov flow. Recall from the start of this chapter that ϕ_t determines two partitions $\mathcal{W}^s = \{W^s(x) : x \in N\}$ and $\mathcal{W}^u = \{W^u(x) : x \in N\}$. These partitions are easily seen to define foliations, and they are known as the *stable* and *unstable* foliations defined by ϕ_t.

We can also define two more foliations $\mathcal{W}^-$ and $\mathcal{W}^+$ associated to the Anosov flow ϕ_t, known as the *weak stable* and w*eak unstable* foliations. Given $x \in N$, define

$$W^-(x) := \bigcup_{t\in\mathbb{R}} \phi_t(W^s(x)) = \bigcup_{t\in\mathbb{R}} W^s(\phi_t x)$$

and

$$W^+(x) := \bigcup_{t\in\mathbb{R}} \phi_t(W^u(x)) = \bigcup_{t\in\mathbb{R}} W^u(\phi_t x).$$

We call $W^-(x)$ and $W^+(x)$ the *weak stable manifold* and the *weak unstable manifold* at x. The foliations $\mathcal{W}^-$ and $\mathcal{W}^+$ are generated by these partitions.

EXERCISE 3.24. Define the *weak stable* and *weak unstable* subbundles $E^-(x) \subseteq T_x N$ and $E^+(x) \subseteq T_x N$ by

$$E^-(x) := E^s(x) \oplus \mathbb{R}X(x),$$
$$E^+(x) := E^u(x) \oplus \mathbb{R}X(x).$$

Use Corollary 3.10 to prove that

$$T_x W^-(x) = E^-(x), \quad T_x W^+(x) = E^+(x).$$

The proof of Theorem 3.17 will require the following theorem of Hirsch, Pugh and Shub [**HPS77**], which we will also need in Chapter 11.

THEOREM 3.25. *Let $\phi_t : N \to N$ be an Anosov flow on a closed 3-manifold N. Then the weak bundles $E^\pm$ are of class C^1.*

We will be most interested in foliations satisfying an additional property.

DEFINITION 3.26. Let N denote a closed 3-manifold which is a circle bundle with fibration $\pi : N \to M$, and let $\mathcal{F}$ denote a foliation on N. We say that π is *transverse to* $\mathcal{F}$ if for all $x \in N$ it holds that

$$T_x N = T_x F(x) \oplus T_x \pi^{-1}(x).$$

The following theorem is due to Ehresmann; the reader may find a proof in [**CN85**, Proposition V.2.1]. In fact, this result holds in a much more general context than is stated below (namely, for *foliated bundles*) but for simplicity we restrict to the case we are interested in.

THEOREM 3.27. *Let N denote a closed 3-manifold which is a circle bundle with fibration $\pi : N \to M$, and let $\mathcal{F}$ denote a foliation transverse to π. Then for each leaf $F(x) \in \mathcal{F}$, the map $\pi|_{F(x)} : F(x) \to M$ is a covering map.*

We now define the *holonomy* of a foliated circle bundle.

DEFINITION 3.28. Let N denote a closed 3-manifold which is a circle bundle with fibration $\pi : N \to M$, and let $\mathcal{F}$ denote a foliation transverse to π.

Fix a loop $f : S^1 \to M$ with $f(0) = f(1) = x$. Suppose $z \in \pi^{-1}(x)$. Since $\pi|_{F(z)}$ is a covering, there exists a unique path $\tilde{f}_z : [0, 1] \to F(z)$ such that $\tilde{f}_z(0) = z$ and $\pi \circ \tilde{f}_z = \tilde{f}^{-1}$, where $\tilde{f}^{-1}$ denotes the path obtained from f by going backwards. Identifying $\pi^{-1}(x) \cong S^1$, this defines a map $\rho(f) : S^1 \to S^1$ by

$$\rho(f)(z) = \tilde{f}_z(1).$$

It is easy to check that $\rho(f)$ depends only on the homotopy class of f, and $\rho(f * g) = \rho(f) \circ \rho(g)$, and hence ρ descends to define a representation $\rho : \pi_1(M, x) \to \mathrm{Diff}(S^1)$. This representation ρ is called the *holonomy* of $(N, M, \pi, \mathcal{F})$.

In fact, we can build a foliation up from a representation.

DEFINITION 3.29. Let M be a closed surface, and let $\rho : \pi_1(M) \to \mathrm{Diff}(S^1)$ be[1] a representation. Let $\tilde{M}$ denote the universal cover of M, and define and action of $\pi_1(M)$ on $\tilde{M} \times S^1$ by

$$\varphi(x, \theta) := (\varphi(x), \rho(\varphi)\theta), \quad \varphi \in \pi_1(M).$$

Note that $\tilde{M} \times S^1$ admits the trivial product foliation $\tilde{\mathcal{F}} = \{\tilde{M} \times \{\theta\} \, : \, \theta \in S^1\}$. Consider the quotient $N := \tilde{M} \times S^1/\pi_1(M)$. The map $\pi = \mathrm{pr}_1 : N \to M$ makes N into circle bundle. Moreover the action of $\pi_1(M)$ obviously preserves the foliation $\tilde{\mathcal{F}}$ of $\tilde{M} \times S^1$, and thus descends to define a foliation $\mathcal{F}$ on N. We refer to the pair $(N, \mathcal{F})$ as a *foliated circle bundle*.

The following result is proved in [**Cal07**, Example 4.2], and shows that foliated circle bundles are essentially the same thing as transverse foliations.

PROPOSITION 3.30. *Let M be a closed surface and $\rho : \pi_1(M) \to \mathrm{Diff}(S^1)$ a representation. Then there exists a closed* 3*-manifold N and a fibration $\pi : N \to M$ with fibre S^1, and a foliation $\mathcal{F}$ of N that is transverse to π, and whose holonomy is precisely ρ. The pair $(N, \mathcal{F})$ is unique up to diffeomorphism. Moreover there is a bijection between isomorphism classes of circle bundles over M and conjugacy classes of representations $\pi_1(M) \to \mathrm{Diff}(S^1)$.*

We can now prove Theorem 3.17.

PROOF. *(of Theorem 3.17)*
The first step in the proof is to show that there exists a closed surface M of genus $\mathrm{g} \geq 2$ and a fibration $\pi : N \to M$ such that the weak stable foliation $\mathcal{W}^+$ of ϕ_t is transverse to π. This is not easy, and we will omit it. The reader is referred to the original argument of Ghys [**Ghy84**, Proposition 2.1]. We remark however that if the foliation $\mathcal{W}^+$ was of class C^2, this result would be an immediate consequence of the main theorem of Thurston's PhD thesis [**Thu72**]. Ghys' proof however does heavily depend on the fact that $\mathcal{W}^+$ is at least of class C^1 (Theorem 3.25). This is not necessarily true in higher dimensions; the proof of Theorem 3.25 hinges on the fact that the foliations $\mathcal{W}^{\pm}$ are of codimension one when N is a 3-manifold. The fact that M necessarily has genus $\mathrm{g} \geq 2$ follows from a theorem due to Plante and Thurston [**PT72**]. Indeed, the fundamental group of a circle bundle N over the sphere or torus has polynomial growth, and the result of Plante and Thurston alluded to above states that if N carries an Anosov flow its fundamental group must grow exponentially.

So let us assume that there exists a closed surface M of genus $\mathrm{g} \geq 2$ and a fibration $\pi : N \to M$ such that $\mathcal{W}^+$ is transverse to π. Let g_0 denote a metric on M of constant negative curvature -1. Then if $\Pi : \tilde{M} \to M$ is the universal cover of M, we have that $\tilde{M} \cong B^2$, $\tilde{M}(\infty) \cong S^1$ and $\Pi : (\tilde{M}, g_{\mathrm{hyp}}) \to (M, g_0)$ is a Riemannian submersion. By Proposition 3.30 we may assume that

$$N = \tilde{M} \times S^1/(x, \theta) \sim (\varphi(x), \rho(\varphi)\theta),$$

where $\rho : \pi_1(M) \to \mathrm{Diff}(S^1)$ is the holonomy of the foliation $\mathcal{W}^+$.

Consider the short exact sequence

$$0 \to \pi_1(S^1) \to \pi_1(N) \to \pi_1(M) \to 0$$

defined by the fibration $S^1 \to N \xrightarrow{\pi} M$. Let Γ denote the image of $\pi_1(S^1)$ in $\pi_1(N)$, and let $\widehat{N}$ be the covering of N corresponding to Γ, i.e. $\widehat{N}/\Gamma = N$. Then the covering $\widehat{N}$ is diffeomorphic to $\tilde{M} \times S^1$. Let $\widehat{\phi}_t$ denote the lift of ϕ_t to a flow on $\widehat{N} = \tilde{M} \times S^1$. Let $p : \tilde{M} \times S^1 \to \tilde{M}$ denote the projection.

EXERCISE 3.31. Show that there exists a constant $P > 0$ such that for each $(x, \theta) \in \tilde{M} \times S^1$, the curve $\gamma_{(x,\theta)}(t) := p \circ \widehat{\phi}_t(x, \theta)$ is a $(P, 0)$-g_{hyp}-quasi-geodesic (see [**Ghy84**, Lemma 3.1] if you get stuck).

Let SM denote the g_0-unit circle bundle of (M, g_0), and let $\psi_t : SM \to SM$ denote the geodesic flow. Let $\tilde{\psi}_t : S\tilde{M} \to S\tilde{M}$ denote the lift of ψ_t to $S\tilde{M} \cong \tilde{M} \times \tilde{M}(\infty)$, so that $\tilde{\psi}_t$ is the geodesic flow of $(\tilde{M}, g_{\mathrm{hyp}})$.

Given a pair $(x, \theta) \in \tilde{M} \times S^1$, the curve $\gamma_{(x,\theta)}$ defined above determines an element $\gamma^*_{(x,v)}(\infty) \in \tilde{M}^*(\infty)$ by Exercise 3.31, and thus by Theorem 3.22 a unique element $\xi_\theta \in \tilde{M}(\infty)$. Let $\xi_\theta^{-1} \in \tilde{M}(\infty)$ denote the element corresponding to $\gamma^*_{(x,\theta)}(-\infty) \in \tilde{M}^*(\infty)$. As the notation implies, the element ξ_θ depends only on θ, not x.

[1]Here $\mathrm{Diff}(S^1)$ denotes the diffeormorphims of the circle that are of class C^1.

Suppose $\zeta, \xi \in \tilde{M}(\infty)$. Then there exists a unique g_{hyp}-geodesic γ such that $\gamma(\infty) = \zeta$ and $\gamma(-\infty) = \xi$. Let $\mathbb{P}(\gamma) : \tilde{M} \to \tilde{M}$ denote orthogonal projection onto γ, and use this to define a map $\mathbb{P}(\zeta, \xi) : \tilde{M} \to S\tilde{M} \cong \tilde{M} \times \tilde{M}(\infty)$ by$\mathbb{P}$

$$\mathbb{P}(\zeta, \xi)(x) = (\gamma(t), \dot{\gamma}(t)) \quad \text{where} \quad \mathbb{P}(\gamma)(x) = \gamma(t).$$

Now define $G_0 : \tilde{M} \times S^1 \to \tilde{M} \times \tilde{M}(\infty)$ by setting

$$G_0(x, \theta) := \mathbb{P}(\xi_\theta, \xi_\theta^{-1})(p(x, \theta)),$$

Then G_0 is continuous and surjective but in general not injective: there may exist two points $(x, \theta), (x', \theta')$ on the same orbit of $\widehat{\phi}_t$ that have the same orthogonal projection onto the g_{hyp}-geodesic γ determined by $\xi_\theta = \xi_{\theta'}$ and $\xi_\theta^{-1} = \xi_{\theta'}^{-1}$. In order to achieve local injectivity we 'average' G_0. For this look at the map $h : \mathbb{R} \times \tilde{M} \times S^1 \to \tilde{M} \times S^1$ defined by

$$G_0(\widehat{\phi}_t(x, \theta)) = \tilde{\psi}_{h(t,(x,\theta))}(G_0(x, \theta)).$$

Then h satisfies the *cocycle property*, that is,

$$h(t + t', (x, \theta)) = h(t, \widehat{\phi}_t(x, \theta)) + h(t', (x, \theta)),$$

as is easily checked. Now choose $\tau \in \mathbb{R}^+$ such that $h(\tau, (x, \theta)) > 0$ for all $(x, \theta) \in \tilde{M} \times S^1$, and then let $r(x, \theta)$ denote the average

$$r(x, \theta) := \frac{1}{\tau} \int_0^\tau h(t, (x, \theta)) dt.$$

Next define $G_\tau : \tilde{M} \times S^1 \to \tilde{M} \times S^1$ by

$$G_\tau(x, \theta) = \tilde{\psi}_{r(x,\theta)}(G_0(x, \theta)).$$

We claim that G_τ is injective on orbits of $\widehat{\phi}_t$. For this observe that if

$$f(t) := r(\widehat{\phi}_t(x, \theta)) + h(t, (x, \theta))$$

then f is monotone increasing. Indeed,

$$\begin{aligned} f'(t) &= \frac{1}{\tau} \int_0^\tau h'(u + t, (x, \theta)) du \\ &= \frac{1}{\tau}(h(\tau + t, (x, \theta)) - h(t, (x, \theta))) \\ &= \frac{1}{\tau} h(\tau, \widehat{\phi}_t(x, \theta)) > 0. \end{aligned}$$

The claim then follows from the computation

$$\begin{aligned} G_\tau(\widehat{\phi}_t(x, \theta)) &= \tilde{\psi}_{r(\tilde{\psi}_t(x,\theta))}(G_0(\widehat{\phi}_t(x, \theta))) \\ &= \tilde{\psi}_{r(\tilde{\psi}_t(x,\theta))+h(t,(x,\theta))}(G_0(x, \theta)) \\ &= \tilde{\psi}_{f(t)}(G_0(x, \theta)). \end{aligned}$$

Thus G_τ is a covering map that intertwines $\widehat{\phi}_t$ and $\tilde{\psi}_t$. Finally, in order to deduce the stronger statement that ϕ_t and some finite covering of ψ_t are also topologically conjugate, one applies the following exercise.

EXERCISE 3.32. By Exercise 3.19 we have

$$SM \cong \tilde{M} \times S^1/(x, \theta) \sim (\varphi(x), \varphi \cdot \theta).$$

Similarly we have

$$N \cong \tilde{M} \times S^1/(x, \theta) \sim (\varphi(x), \rho(\varphi)\theta).$$

Show that G_τ is equivariant under these two actions of $\pi_1(M)$, and thus G_τ descends to define a finite covering map G'_τ that intertwines ϕ_t and ψ_t. Complete the details of the proof above by showing that G_0 is surjective and by studying in detail where the injectivity fails.

This completes the proof. □

Here is a easy corollary of Theorem 3.17.

COROLLARY 3.33. *Let N be a closed 3-manifold which is a circle bundle. Suppose $\phi_t : N \to N$ is an Anosov flow. Then ϕ_t is transitive, and the non-wandering set Ω is equal to all of N.*

PROOF. The geodesic flow is transitive by Corollary 3.14 and 3.12. Since orbit equivalence clearly preserves transitivity, Theorem 3.17 completes the proof. □

REMARK 3.34. In order to show why this theorem is useful, let us remark that there exist so-called 'anomalous Anosov flows' constructed by Franks and Williams [**FW80**] on closed 3-manifolds whose non-wandering set is not the entire manifold. Combining Theorem 3.17 with Corollary 3.14 we see that such a manifold *cannot be a circle bundle*.

CHAPTER 4

Surface theory

In this chapter we specialize to the case that we are most interested in: the case of a closed surface, that is, a closed 2-dimensional Riemannian manifold. In the first section we present a few standard results in two-dimensional Riemannian geometry on the structure of the unit sphere bundle of a surface. In particular we construct a moving frame $\{X, H, V\}$ of SM and derive Cartan's structural equations. Then in the second section of the chapter we derive the Jacobi equations of the geodesic flow.

4.1. The structure of the tangent bundle of a closed surface

Let (M, g) be a closed Riemannian surface with geodesic flow $\phi_t : SM \to SM$. Let $\pi : TM \to M$ denote the footprint map (we shall often consider π as a map $SM \to M$) and ∇ the Levi-Civita connection on M.

DEFINITION 4.1. Let $\mathcal{K} : T(TM) \to TM$ denote the *connection map* of ∇, defined as follows. If $\xi \in T_{(x,v)}TM$, let $c : (-\epsilon, \epsilon) \to TM$ denote a curve adapted to ξ. Write

$$c(t) = (\gamma(t), Z(t)),$$

so $Z(t)$ is a vector field along $\gamma(t)$, and define

$$\mathcal{K}_{(x,v)}(\xi) := \frac{DZ}{dt}(0) \in T_x M,$$

where $\frac{D}{dt}$ is the covariant derivative of ∇.

We can define a complementary map $\mathcal{L} : TM \to T(TM)$ as follows. Given $(x, v) \in TM$ and $w \in T_x M$, let $\gamma : (-\epsilon, \epsilon) \to M$ denote a curve adapted to (x, w) and let $Z(t)$ denote the parallel transport of v along γ. Then we can consider the curve

$$c(t) := (\gamma(t), Z(t)) : (-\epsilon, \epsilon) \to TM.$$

Define

$$\mathcal{L}_{(x,v)}(w) := \dot{c}(0) \in T_{(x,v)}TM.$$

Then one easily checks

$$\mathcal{K} \circ \mathcal{L} = 0,$$

$$d\pi \circ \mathcal{L} = \mathrm{Id},$$

and hence we obtain a splitting

$$T_{(x,v)}TM = \mathbb{H}(x, v) \oplus \tilde{\mathbb{V}}(x, v),$$

where

$$\mathbb{H}(x, v) := \ker \left\{ \mathcal{K}_{(x,v)} : T_{(x,v)}TM \to T_x M \right\},$$

and

$$\tilde{\mathbb{V}}(x, v) := \ker \left\{ d_{(x,v)}\pi : T_{(x,v)}TM \to T_x M \right\},$$

with $d\pi : \mathbb{H}(x, v) \to T_x M$ and $\mathcal{K} : \tilde{\mathbb{V}}(x, v) \to T_x M$ linear isomorphisms.

Restricting to SM we obtain a similar splitting

$$T_{(x,v)}SM = \mathbb{H}(x, v) \oplus \mathbb{V}(x, v), \tag{4.1.1}$$

where now

$$\mathbb{V}(x, v) := \ker \left\{ d_{(x,v)}\pi(x, v) : T_{(x,v)}SM \to T_x M \right\}.$$

DEFINITION 4.2. This decomposition allows us to define the *Sasaki metric* $\langle\langle\cdot,\cdot\rangle\rangle$ on TM by setting

$$\langle\langle\xi,\eta\rangle\rangle := \langle\mathcal{K}(\xi),\mathcal{K}(\eta)\rangle + \langle d\pi(\xi), d\pi(\eta)\rangle\,;$$

note that this makes the decomposition (4.1.1) an orthogonal one. We will generally use the restriction of the Sasaki metric to SM, and will use this metric on SM without further comment.

LEMMA 4.3. *The* 2*-form* $d\alpha \in \Omega^2(TM)$ *can alternatively be defined by*

$$d\alpha(\xi,\eta) = \langle\mathcal{K}(\xi), d\pi(\eta)\rangle - \langle\mathcal{K}(\eta), d\pi(\xi)\rangle\,.$$

PROOF. Let $C : (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon) \to TM$ denote a smooth map such that $C(0,0) = (x,v)$ and such that

$$\frac{\partial}{\partial s}\Big|_{s=0} C(s,0) = \xi, \quad \frac{\partial}{\partial t}\Big|_{t=0} C(0,t) = \eta,$$

where $\xi,\eta \in T_{(x,v)}TM$.

Then

$$\begin{aligned}
d\alpha_{(x,v)}(\xi,\eta) &= \frac{\partial}{\partial s}\Big|_{s=0}\alpha\left(\frac{\partial}{\partial t}\Big|_{t=0}C(s,t)\right) - \frac{\partial}{\partial t}\Big|_{t=0}\alpha\left(\frac{\partial}{\partial s}\Big|_{s=0}C(s,t)\right)\\
&= \frac{\partial}{\partial s}\Big|_{s=0}\left\langle C(s,0), \frac{\partial}{\partial t}\Big|_{t=0}\pi\circ C(s,t)\right\rangle - \frac{\partial}{\partial t}\Big|_{t=0}\left\langle C(0,t), \frac{\partial}{\partial s}\Big|_{s=0}\pi\circ C(s,t)\right\rangle\\
&\overset{(*)}{=} \left\langle \nabla_{\frac{\partial}{\partial s}} C(s,0), \frac{\partial}{\partial t}\pi\circ C(s,t)\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial t}} C(0,t), \frac{\partial}{\partial s}\pi\circ C(s,t)\right\rangle\Big|_{s=t=0}\\
&= \langle\mathcal{K}(\xi), d\pi(\eta)\rangle - \langle\mathcal{K}(\eta), d\pi(\xi)\rangle\,,
\end{aligned}$$

where $(*)$ used the fact that ∇ is compatible with the metric and that it is symmetric. □

Since M is an oriented 2-manifold, each tangent space $T_xM \cong \mathbb{R}^2$ is an oriented vector space'. Thus given a unit vector $v \in T_xM$, we write $e^{it}v$ for vector obtained by rotating v by an angle t according to the orientation of the surface. Thus $e^{i\pi/2}v$ is the unique vector such that $\{v, e^{i\pi/2}v\}$ is a positively oriented orthonormal basis of T_xM. Let $\mathtt{i} : TM \to TM$ denote the almost complex structure on M defined by

$$\mathtt{i}(x,v) = (x, e^{i\pi/2}v). \tag{4.1.2}$$

Given a point $v \in T_xM$, we will often abuse notation slightly and simply write $\mathtt{i}v$ for the vector $e^{i\pi/2}v$.

DEFINITION 4.4. Define a flow $\rho_t : SM \to SM$ by

$$\rho_t(x,v) = (x, e^{it}v),$$

and let

$$V(x,v) := \frac{d}{dt}\Big|_{t=0}\rho_t(x,v)$$

denote the infinitesimal generator of this flow.

We see that

$$d\pi(V(x,v)) = d\pi\left(\frac{d}{dt}\Big|_{t=0}\rho_t(x,v)\right) = 0,$$

since $\pi\circ\rho_t$ is the constant map $S_xM \to \{x\}$.

Moreover

$$\begin{aligned}
\mathcal{K}(V(x,v)) &= \mathcal{K}\left(\frac{d}{dt}\Big|_{t=0}\rho_t(x,v)\right)\\
&= \frac{d}{dt}\Big|_{t=0}e^{it}v\\
&= \mathtt{i}v.
\end{aligned}$$

In particular this shows that the one dimensional distribution $\mathbb{V}$ of TSM is spanned by V. We call V the *vertical vector field* on SM.

DEFINITION 4.5. We define the *connection* 1*-form* ψ to be the dual 1-form to V via the Sasaki metric, that is

$$\psi_{(x,v)}(\xi) := \langle\langle V(x,v),\xi\rangle\rangle\,.$$

EXERCISE 4.6. Show that

$$\psi_{(x,v)}(\xi) = \langle \mathcal{K}(\xi), \mathtt{i}v \rangle .$$

Next, let α denote the contact form from Definition 1.23. Show that α is the 1-form dual to the geodesic vector field X under $\langle\langle \cdot, \cdot \rangle\rangle$. Finally define a third 1-form β by

$$\beta_{(x,v)}(\xi) = \langle d\pi(\xi), \mathtt{i}v \rangle .$$

Prove that $\{\alpha, \beta, \psi\}$ is a *moving coframe* for SM. That is, for all $(x, v) \in SM$, $\{\alpha_{(x,v)}, \beta_{(x,v)}, \psi_{(x,v)}\}$ is a basis of $T^*_{(x,v)}SM$.

DEFINITION 4.7. Let H denote the vector field dual to β. We call H the *horizontal vector field*.

Summarizing, we have defined a moving frame $\{X, H, V\}$ of TSM with dual moving coframe $\{\alpha, \beta, \psi\}$ of T^*SM. The next thing to observe is the *Cartan structure equations*; these will be very useful for us.

Fix a point $x_0 \in M$. There exist *isothermal coordinates* (x^1, x^2) on a neighborhood U about x_0, that is, the length element ds^2 of g is written as

$$ds^2 = e^{2\lambda(x^1,x^2)} \left\{ \left(dx^1\right)^2 + \left(dx^2\right)^2 \right\}$$

for some smooth function λ (see [**Spi99**, Volume IV, p314]) . If (x^1, x^2, v_1, v_2) are the induced local coordinates on $TM|_U$ then we have

$$SM|_U = \left\{ (x, v) \in TM|_U \, : \, e^{2\lambda} \left\{ (v^1)^2 + (v^2)^2 \right\} = 1 \right\}.$$

Thus we can define coordinates (x^1, x^2, θ) on $SM|U$ via

$$(x^1, x^2, \theta) \mapsto (x^1, x^2, e^{-\lambda}\cos\theta, e^{-\lambda}\sin\theta) \in SM|_U.$$

We now wish to compute the vector fields X, H, V in terms of these local coordinates.

Let $\{\Gamma^k_{ij}\}$ denote the Christoffel symbols of ∇. We can write the geodesic vector field X in the local coordinates (x^1, x^2, v_1, v_2) on $TM|_U$ as

$$X(x, v) = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x) v^j v^k \frac{\partial}{\partial v^i} \tag{4.1.3}$$

(see [**GHL04**, Section 2.219]). Recalling that the Christoffel symbols Γ^i_{jk} are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \left(\partial_j g_{k\ell} + \partial_k g_{\ell j} - \partial_\ell g_{jk} \right)$$

(see [**GHL04**, Proposition 2.54]) we compute in our isothermal coordinates (x^1, x^2) that

$$\Gamma^i_{jk} = \partial_j \lambda \delta^i_k + \partial_k \lambda \delta^i_j - \partial_i \lambda \delta_{jk}.$$

From this we can compute $X(x, v)$ using (4.1.3) in terms of our coordinates (x^1, x^2, v_1, v_2); then using

$$\frac{\partial v^i}{\partial \theta} = -e^{-\lambda} \sin\theta, \quad \frac{\partial v^2}{\partial \theta} = e^{-\lambda} \cos\theta,$$

we can compute $X(x, v)$ in terms of our local coordinates $\left(x^1, x^2, \theta\right)$ to obtain

$$X\,(x, v) = e^{-\lambda} \left(\cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial x^2} + \left\{ -\frac{\partial \lambda}{\partial x^1} \sin\theta + \frac{\partial \lambda}{\partial x^2} \cos\theta \right\} \frac{\partial}{\partial \theta} \right). \tag{4.1.4}$$

EXERCISE 4.8. Verify this formula. Moreover, prove that

$$H\,(x, v) = e^{-\lambda} \left(-\sin\theta \frac{\partial}{\partial x^1} + \cos\theta \frac{\partial}{\partial x^2} - \left\{ \frac{\partial \lambda}{\partial x^1} \cos\theta + \frac{\partial \lambda}{\partial x^2} \sin\theta \right\} \frac{\partial}{\partial \theta} \right) \tag{4.1.5}$$

(it may be helpful to use the following alternative (though equivalent) description of H. Given $(x, v) \in SM$, the flow h_t of H is given by

$$h_t(x, v) = (\gamma_{(x,\mathtt{i}v)}(t), Z(t)),$$

where $Z(t)$ is the parallel transport of v along $\gamma_{(x,\mathtt{i}v)}(t)$). Finally, show that V is given simply by

$$V(x, v) = \frac{\partial}{\partial \theta}.$$

From these three relations we can deduce Cartan's structural equations for the commutators $[H, V]$, $[V, X]$ and $[X, H]$. Recall that the Gaussian curvature K of M can be given in the isothermal coordinates (x^1, x^2) by

$$K = -e^{-2\lambda}\Delta\lambda$$

(see [**Spi99**, Volume III, p136]). Easy calculations then give

$$\begin{aligned} [H, V] &= X, \\ [V, X] &= H \\ [X, H] &= KV. \end{aligned} \tag{4.1.6}$$

REMARK 4.9. Note the similarity of these and the equations (3.1.2) in Example 3.5; indeed (3.1.2) corresponds to the case $K = -1$ (see Remark 3.7).

EXERCISE 4.10. Deduce the following structure equations for the dual coframe $\{\alpha, \beta, \psi\}$:

$$\begin{aligned} d\alpha &= \psi \wedge \beta, \\ d\beta &= -\psi \wedge \alpha, \\ d\psi &= -K\alpha \wedge \beta. \end{aligned} \tag{4.1.7}$$

The reader is referred to [**ST67**, Section 7.2] for an alternative derivation of the structural equations not requiring the use of isothermal coordinates.

DEFINITION 4.11. Let us define the *area form* Ω_a on M by

$$\Omega_a(v, u) := \langle u, \mathtt{i}v \rangle ,$$

for any $u, v \in T_xM$, so for $v \in S_xM$,

$$\Omega_a(v, \mathtt{i}v) = 1.$$

Then clearly

$$\pi^*\Omega_a = \alpha \wedge \beta, \tag{4.1.8}$$

since

$$\begin{aligned} \pi^*\Omega_a(\xi, \eta) &= \Omega_a(d\pi(\xi), d\pi(\eta)) \\ &= \langle d\pi(\eta), \mathtt{i}d\pi(\xi)\rangle \\ &= \langle d\pi(\eta), \mathtt{i}\alpha(\xi)v + \mathtt{i}\beta(\xi)\mathtt{i}v\rangle \\ &= \alpha(\xi)\beta(\eta) - \beta(\xi)\alpha(\eta) \\ &= \alpha \wedge \beta(\xi, \eta). \end{aligned}$$

Thus using (4.1.7) we have also shown

$$d\psi = -K\pi^*\Omega_a,$$

which can be used to *define* the Gaussian curvature (see the aforementioned [**ST67**, Section 7.2]).

SUMMARY 4.12. It may be helpful to the reader to summarize the results of this section. We have shown that if we define 1-forms α, β, ψ on SM by

$$\begin{aligned} \alpha_{(x,v)}(\xi) &= \langle d\pi\,(\xi)\,, v\rangle\,, \\ \beta_{(x,v)}(\xi) &= \langle d\pi(\xi), \mathtt{i}v\rangle\,, \\ \psi_{(x,v)}(\xi) &= \langle \mathcal{K}(\xi), \mathtt{i}v\rangle\,, \end{aligned}$$

then $\{\alpha, \beta, \psi\}$ is a smooth coframe of SM and satisfy the structural equations 4.1.7, and if Ω_a denotes the area form then

$$d\psi = -K\pi^*\Omega_a.$$

Moreover if $\{X, H, V\}$ denotes the dual frame then X is the geodesic vector field

$$X(x, v) = \frac{d}{dt}\Big|_{t=0}\phi_t(x, v)$$

where $\phi_t : SM \to SM$ is the geodesic flow, and X, H and V satisfy the structural equations 4.1.6.

4.2. The Jacobi equations

We now wish to obtain ODE's for the evolution of the differential of ϕ_t. In order to do so, we need to make the following remark, which we shall use many times throughout these notes.

REMARK 4.13. Suppose Y and Z are two vector fields on a manifold N, and ψ_t denotes the local flow of Y. Then by definition,

$$[Y, Z](x) = \frac{d}{dt}\Big|_{t=0} d\psi_{-t}(Z(\psi_t x)),$$

and hence

$$d\psi_{-t}\{[Y, Z](\psi_t x))\} = \frac{d}{dt}\{d\psi_{-t}(Z(\psi_t x))\}. \tag{4.2.1}$$

Let M be an arbitrary surface, and ϕ_t the geodesic flow on M. Fix a point $(x_0, v_0) \in SM$. We will adopt the following notation: let $H(t) := H(\phi_t(x_0, v_0))$ and $H = H(0) = H(x, v_0)$, and similarly for $X(t), V(t)$ etc.

Let $\xi \in T_{(x_0,v_0)}SM$. We can write

$$\xi = aX + yH + zV$$

for some constants $a, y, z \in \mathbb{R}$. Moreover there exist smooth functions $a(t), b(t), c(t)$ satisfying

$$d\phi_t(\xi) = a(t)X(t) + y(t)H(t) + z(t)V(t), \tag{4.2.2}$$

subject to the initial conditions

$$\begin{aligned} a(0) &= a, \\ y(0) &= y, \\ z(0) &= z. \end{aligned} \tag{4.2.3}$$

We will derive ODE's that $a(t), y(t), z(t)$ must satisfy. These are Jacobi-type equations; for the purposes of this course we shall refer to them as 'the' *Jacobi equations*.

PROPOSITION 4.14. *The functions $a(t), y(t), z(t)$ satisfying* (4.2.2) *and* (4.2.3) *satisfy the Jacobi equations:*

$$\begin{aligned} \dot{a} &= 0, \\ \dot{y} - z &= 0, \\ \dot{z} + Ky &= 0. \end{aligned} \tag{4.2.4}$$

PROOF. We begin by applying $d\phi_{-t}$ to both sides of (4.2.2) to obtain

$$\xi = a(t)d\phi_{-t}(X(t)) + y(t)d\phi_{-t}(H(t)) + z(t)d\phi_{-t}(V(t)),$$

and then differentiating both sides with respect to t and applying (4.2.1) we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}(\xi) \\ &= \dot{a}(t)d\phi_{-t}(X(t)) + a(t)d\phi_{-t}([X, X](t)) + \dot{y}(t)d\phi_{-t}(H(t)) \\ &\quad + y(t)d\phi_{-t}([X, H](t)) + \dot{z}(t)d\phi_{-t}(V(t)) + z(t)d\phi_{-t}([X, V](t)), \end{aligned}$$

and then applying the structure equations and grouping like terms we obtain

$$0 = d\phi_{-t}\{\dot{a}(t)X(t) + (\dot{y}(t) - z(t))H(t) + (\dot{z}(t) + K(t)y(t))V(t)\}.$$

Since $d\phi_{-t}$ is an isomorphism (being invertible) and $\{X(t), H(t), V(t)\}$ is a basis of each tangent space $T_{\phi_t(x_0,v_0)}SM$ the coefficients of $X(t)$, $H(t)$ and $V(t)$ must vanish for all t, and this is precisely what we wanted to show. □

REMARK 4.15. Note that if $K \equiv -1$ then the Jacobi equations have the solutions $y(t) = e^{\pm t}$, and we recover the result of Lemma 3.6.

CHAPTER 5

A criterion for a flow to be Anosov

The aim of this chapter is to prove a general criterion for a flow $\phi_t : N \to N$ on a closed manifold N to be Anosov. As a corollary to this we will be able to prove Theorem 1.6. In the second subsection we will go through the explicit construction due to V. J. Donnay and C. C. Pugh [**DP03**] of embedded closed surfaces in $\mathbb{R}^3$ whose geodesic flows are Anosov. These examples contain, of course, regions with positive curvature.

5.1. The criterion

DEFINITION 5.1. Here is the general setting. Let N be a closed manifold, and ϕ_t a *non-singular* flow on N with infinitesimal generator X.

We consider the *quotient bundle* $\widehat{T}N$ defined by

$$\widehat{T}_x N := T_x N/\mathbb{R}X(x).$$

Since $d\phi_t(X(x)) = X(\phi_t x)$, $d\phi_t$ descends to the quotient to define a map $A_t : \widehat{T}_x N \to \widehat{T}_{\phi_t x} N$ satisfying

$$A_{s+t} = A_s \circ A_t.$$

The following theorem is the main result of this chapter, and was proved by Wojtkowski in [**Woj00**, Theorem 5.2], and extends to flows an earlier result of Lewowicz [**Lew80**] for diffeomorphisms.

THEOREM 5.2. *Let N be a closed manifold and $\phi_t : N \to N$ a non-singular flow on N with infinitesimal generator X. Suppose there exists a quadratic form $Q : TN \to \mathbb{R}$ on TN satisfying the following four properties:*

(1) *For each $x \in N$, the form $Q_x := Q|_{T_x N} : T_x N \to \mathbb{R}$ depends continuously on x.*
(2) *For all $x \in N$, $v \in T_x N$ and $a \in \mathbb{R}$, we have*

$$Q_x(v + aX(x)) = Q_x(v).$$

Thus Q can be projected onto the quotient bundle $\widehat{T}N$ to define

$$\widehat{Q} : \widehat{T}N \to \mathbb{R}.$$

(3) *$\widehat{Q} : \widehat{T}N \to \mathbb{R}$ is non-degenerate.*
(4) *The Lie derivative $L_X Q$ must be continuous, and if $\widehat{L}$ denotes the projection of the Lie derivative $L_X Q$ to $\widehat{T}N$, so*

$$\widehat{L}(v) := \frac{d}{dt}\Big|_{t=0} \widehat{Q}(A_t(v)),$$

then $\widehat{L}$ must be positive definite on $\widehat{T}N$.

Then the flow ϕ_t is Anosov.

The proof of this theorem will take some time; before we begin with the proof however we will deduce Theorem 1.6.

PROOF. *(of Theorem* 1.6)

We simply need to exhibit a quadratic form Q on TSM satisfying the four conditions of Theorem 5.2. Given $\xi \in T_{(x,v)}SM$ we may write

$$\xi = aX(x,v) + yH(x,v) + zV(x,v),$$

for some constants a, y, z. We define our quadratic form Q by

$$Q_{(x,v)}(\xi) := yz. \tag{5.1.1}$$

Checking the first three conditions of Theorem 5.2 is trivial, but the fourth requires the Jacobi equations (4.2.4). Adopting the notation of Proposition 4.14, we have

$$\begin{aligned}\widehat{L}(\xi) &= \frac{d}{dt}\Big|_{t=0}\hat{Q}(A_t(\xi)) \\ &= \frac{d}{dt}\Big|_{t=0}y(t)z(t) \\ &= \dot{y}(0)z(0)+y(0)\dot{z}(0) \\ &= z^2-K(x,v)y^2 \\ &> 0,\end{aligned}$$

if $\xi - aX \neq 0$, since $K(x,v) < 0$. □

An important step in proving Theorem 5.2 is the following auxiliary proposition.

PROPOSITION 5.3. *Let N be a closed manifold, and ϕ_t a non-singular flow on N with infinitesimal generator X. The flow ϕ_t is Anosov if and only if there exists an A_t-invariant splitting of $\widehat{T}N$ as*

$$\widehat{T}N = \widehat{E}^s \oplus \widehat{E}^u,$$

and constants $C, \mu > 0$ such that for all $t \geq 0$,

$$\left\| A_t|_{\widehat{E}^s} \right\| \leq Ce^{-\mu t},$$

$$\left\| A_{-t}|_{\widehat{E}^u} \right\| \leq Ce^{-\mu t}.$$

This proposition is easy if ϕ_t has a codimension 1 invariant subbundle, which is the case for the geodesic flow. This is not true in general though.

PROOF. One direction is trivial - if ϕ_t is Anosov then the desired A_t-splitting arises from the $d\phi_t$-splitting; we simply identify $\widehat{T}N$ with $E^s \oplus E^u$. For the converse, we need to know how to reconstruct E^s and E^u from knowledge of $\widehat{E}^s$ and $\widehat{E}^u$, and this is not entirely obvious.

Since $X(x) \neq 0$ for all $x \in N$ we can choose a Riemannian metric on N such that $|X(x)| = 1$ for all $x \in N$. Under this metric we can identify $\widehat{T}N$ with $(\mathbb{R}X)^{\perp}$, which we will do from now on without further comment.

Here is the plan: we will find functions $s : \widehat{E}^s \to \mathbb{R}, u : \widehat{E}^u \to \mathbb{R}$ that are linear in v and such that if we set

$$E^s(x) := \left\{ v + s(x,v)X(x) \,:\, v \in \widehat{E}^s \right\};$$

$$E^u(x) := \left\{ v + u(x,v)X(x) \,:\, v \in \widehat{E}^u \right\},$$

then E^s and E^u are $d\phi_t$-invariant and satisfy the required decay properties for the Anosov splitting.

Now since

$$d\phi_t(v) = A_t(v) + \langle d\phi_t(v), X(\phi_t x)\rangle\, X(\phi_t x), \tag{5.1.2}$$

if such a function $s(x,v)$ exists, then

$$\begin{aligned} d\phi_t(v + s(x,v)X(x)) &= d\phi_t(v) + s(x,v)X(\phi_t x) \\ &= A_t(v) + (s(x,v) + \langle d\phi_t(v), X(\phi_t x)\rangle)\, X(\phi_t x),\end{aligned}$$

and hence for our proposed definition of E^s to be $d\phi_t$-invariant we need

$$s(\phi_t x, A_t(v)) = s(x,v) + \langle d\phi_t(v), X(\phi_t x)\rangle\,. \tag{5.1.3}$$

Now define

$$S(x,v) := -\frac{d}{dt}\Big|_{t=0}\langle d\phi_t(v), X(\phi_t x)\rangle\,.$$

Note that $S(x,v)$ is linear in v, and that

$$S(\phi_t x, A_t(v)) = -\frac{d}{dt}\,\langle d\phi_t(v), X(\phi_t x)\rangle\,.$$

Indeed,

$$\begin{aligned} S(\phi_t x, A_t(v)) &= -\frac{d}{dr}\Big|_{r=0}\langle d\phi_r(A_t(v)), X(\phi_r(\phi_t x))\rangle \\ &= -\frac{d}{dr}\Big|_{r=0}\langle d\phi_{r+t}(v), X(\phi_{r+t}x)\rangle \\ &= -\frac{d}{dr}\Big|_{r=t}\langle d\phi_r(v), X(\phi_r x)\rangle . \end{aligned}$$

Now define

$$s(x, v) := \int_0^\infty S(\phi_r x, A_r(v))dr.$$

Since S is linear in v and $|A_r(v)|$ decreases exponentially as $r \to \infty$ for $v \in \widehat{E}^s$, the above integral converges uniformly, and thus s is well defined. Next, observe that

$$\begin{aligned} s(\phi_t x, A_t(v)) &= \int_0^\infty S(\phi_{r+t}x, A_{r+t}(v))dr \\ &= \int_t^\infty S(\phi_r x, A_r(v))dr \\ &= s(x, v) - \int_0^t S(\phi_r x, A_r(v))dr \\ &= s(x, v) + \langle d\phi_r(v), X(\phi_r x)\rangle \Big|_{r=0}^{r=t} \\ &= s(x, v) + \langle d\phi_t(v), X(\phi_t x)\rangle , \end{aligned}$$

since $v \in \widehat{E}^s \subseteq (\mathbb{R}X)^\perp$, and thus (5.1.3) is verified. This gives us our desired subbundle E^s . Since $s(x, v)$ is linear in v, (5.1.2) implies we can find constants $C, \mu \geq 0$ such that

$$|d\phi_t(v)| \leq Ce^{-\mu t}|v| \quad \text{for all } t \geq 0 \text{ and } v \in E^s.$$

The construction for the unstable bundle is similar, and is left as an exercise:

EXERCISE 5.4. Complete the proof of Proposition 5.3 by constructing the unstable subbundle E^u. Use the same ideas to show that the Anosov property is invariant under time changes, in other words, if the flow of X is Anosov, then the flow of fX is also Anosov, where f is any smooth positive function.

□

We will now prove Theorem 5.2.

PROOF. *(of Theorem 5.2)*

Due to the previous proposition it is enough to exhibit a splitting $\widehat{T}N = \widehat{E}^s \oplus \widehat{E}^u$ satisfying the hypotheses of Proposition 5.3. Set, for $x \in N$:

$$\begin{aligned} C^+(x) &:= \left\{ v \in \widehat{T}_x N \ : \ \widehat{Q}_x(v) \geq 0 \right\}, \\ C^-(x) &:= \left\{ v \in \widehat{T}_x N \ : \ \widehat{Q}_x(v) \leq 0 \right\}, \\ C^+_\infty(x) &:= \bigcap_{t \geq 0} A_t(C^+(\phi_{-t}x)), \\ C^-_\infty(x) &:= \bigcap_{t \geq 0} A_{-t}(C^-(\phi_t x)), \end{aligned}$$

and then set $C^+ := \coprod_{x \in N} C^+(x)$ etc. We will now use the hypotheses of the theorem to prove the existence of five constants $\alpha, \beta, \gamma, \delta, \varepsilon > 0$.

$\underline{\alpha}$: By compactness and the fact that $\widehat{L}$ is positive definite we deduce the existence of $\alpha > 0$ such that

$$\widehat{L}(v) \geq \alpha\, |v|^2 \quad \text{for all } \ v \in \widehat{T}N.$$

$\underline{\beta}$: By compactness there exists $\beta > 0$ such that

$$\left|\widehat{Q}(v)\right| \leq \beta\, |v|^2 \quad \text{for all } \ v \in \widehat{T}N.$$

Thus for all $v \in \widehat{T}N$ and $t > 0$ we have

$$\begin{aligned} \frac{d}{dt}\widehat{Q}(A_t(v)) &\geq \alpha\,|A_t(v)|^2 \\ &\geq \frac{\alpha}{\beta}\left|\widehat{Q}(A_t(v))\right|. \end{aligned} \tag{5.1.4}$$

In particular for all $t > 0$ we have

$$A_t(C^+) \subset \operatorname{int}(C^+) \cup \{0\},$$
$$A_{-t}(C^-) \subset \operatorname{int}(C^-) \cup \{0\}.$$

$\underline{\gamma}$: Now we integrate (5.1.4) with respect to t to discover that if $\gamma := \frac{\alpha}{\beta}$ then

$$\frac{\widehat{Q}(A_t(v))}{\widehat{Q}(v)} \geq e^{\gamma t} \quad \text{for} \quad v \in \operatorname{int}(C^+),$$

and

$$\frac{\widehat{Q}(A_t(v))}{\widehat{Q}(v)} \leq e^{-\gamma t} \;\; \text{for} \;\; A_t v \in \operatorname{int}(C^-).$$

$\underline{\delta}$: By compactness there exists $\delta > 0$ such that for all $v \in A_1(C^+)$ (and thus in particular for all $v \in C_\infty^+$),

$$\widehat{Q}(v) \geq \delta\,|v|^2.$$

$\underline{\varepsilon}$: By compactness there exists $\varepsilon > 0$ such that for all $v \in A_{-1}(C^-)$ (and thus in particular for all $v \in C_\infty^-$),

$$-\widehat{Q}(v) \geq \varepsilon\,|v|^2.$$

The next part of the proof is the following claim:

Claim: If $\widehat{Q}$ has (constant) signature (ℓ, m) with $l + m = \dim N - 1$, then for all $x \in N$, $C_\infty^+(x)$ contains an ℓ-dimensional subspace and $C_\infty^-(x)$ contains an m-dimensional subspace.

Proof of claim: Let $G_\ell(x)$ denote the compact Grassmannian of ℓ-planes in $\widehat{T}_x N$, and let L_n denote the set of points in $G_\ell(x)$ corresponding to ℓ-planes in $A_n(C^+(\phi_{-n}x))$. Then each L_n is a closed non-empty set in $G_\ell(x)$ with $L_{n+1} \subseteq L_n$. Thus compactness implies $\bigcap_{n\in\mathbb{N}} L_n \neq \emptyset$, which proves the claim for $C_\infty^+(x)$. The proof is similar for $C_\infty^-(x)$.

It is now easy to complete the proof. If $v \in C_\infty^+(x)$ and $t > 0$ we have

$$\begin{aligned} \beta\,|v|^2 &\geq \widehat{Q}(v) \\ &= \widehat{Q}(A_t(A_{-t}(v))) \\ &\geq e^{\gamma t}\widehat{Q}(A_{-t}(v)) \\ &\geq \delta e^{\gamma t}\,|A_{-t}(v)|^2, \end{aligned}$$

and thus

$$|A_{-t}(v)| \leq \sqrt{\frac{\beta}{\delta}} e^{-\frac{\gamma t}{2}}\,|v|.$$

Similarly if $v \in C_\infty^-(x)$ we obtain

$$|A_t(v)| \leq \sqrt{\frac{\beta}{\varepsilon}} e^{-\frac{\gamma t}{2}}\,|v|.$$

Thus if

$$\widehat{E}^s(x) := \operatorname{span}\{C_\infty^-(x)\}$$

and

$$\widehat{E}^u(x) := \operatorname{span}\left\{C_\infty^+(x)\right\}$$

then we can find constants $C, \mu > 0$ such that $\widehat{E}^s$ and $\widehat{E}^u$ satisfy the decay conditions of Proposition 5.3. Finally, one observes that $T\widehat{N} = \widehat{E}^s \oplus \widehat{E}^u$. Indeed, we must have $\widehat{E}^s \cap \widehat{E}^u = \{0\}$, since a non-zero vector in the intersection would have contradictory decay properties. Moreover since $\dim\, T_x\widehat{N} = l + m$, the two subbundles span $T\widehat{N}$. This completes the proof. $\square$

REMARK 5.5. In fact, the converse to the theorem above also holds. Namely, if ϕ_t is Anosov, then there exists a quadratic form Q satisfying the hypotheses of Theorem 5.2. This is proved in [**Lew80**] for diffeomorphisms, and the proof for flows is similar. As an application of the converse of Theorem 5.2, one can deduce the following result.

COROLLARY 5.6. *The set of Anosov flows is C^1-open.*

We conclude this section with a couple of exercises.

EXERCISE 5.7. This exercise is one step in the proof of the converse to Theorem 5.2. Show that if ϕ_t is an Anosov flow with infinitesimal generator X and there exists a quadratic form Q satisfying conditions (1) and (2) and (4) of Theorem 5.2 then in fact Q also satisfies condition (3), that is, the projected form $\widehat{Q}$ is non-degenerate.

EXERCISE 5.8. Let (M, g) be a closed surface with curvature K. Suppose that $K \leq 0$, and that $K = 0$ at only finitely many points. Prove that the geodesic flow of (M, g) is Anosov.

5.2. The Donnay-Pugh example

The purpose of this section is to sketch the construction of V.J. Donnay and C.C. Pugh [**DP03**] of embedded closed surfaces in $\mathbb{R}^3$ whose geodesic flows are Anosov.

DEFINITION 5.9. A *dispersing tube* is a surface of revolution in $\mathbb{R}^3$ given in cylindrical coordinates (r, θ, z) by a function $r = h(z)$ such that $h : [-1, 1] \to (0, 1]$ satisfies:

(1) $h(z) = h(-z)$;
(2) $h(\pm 1) = 1$;
(3) if $|z| < 1$, then h is smooth and $h''(z) > 0$;
(4) the graph of h has infinite order contact with the lines $z = \pm 1$. In particular $\lim_{z \to \pm 1} h'(z) = \pm\infty$.

The surface T looks like a catenoid and it has planar ends and negative curvature. Its geodesics are easy to describe. There is the closed geodesic Γ around the waist of the tube ($z = 0$) and there are geodesics asymptotic to it. Every other geodesic either enters and exits T without meeting Γ, or it crosses Γ once on its way from one end of the tube to the other. The entry and exit angles are equal because the tube is symmetric.

Note that we can perform independent linear scalings in the variables z and r without altering the main properties of T.

We will now show how to get the Anosov condition. We start with the following exercise.

EXERCISE 5.10. Let M be a closed surface. Given a Riemannian metric g on M, let K_g denote the sectional curvature of (M, g), and let $\phi_t^g : S^g M \to S^g M$ denote the geodesic flow of (M, g). Show that if $|K_g| \leq 2$, there exist positive constants a and b *independent* of g such that for all $t \in [0, 1]$ and all ξ,

$$a|\xi|^2 \leq |d\phi_t^g(\xi)|^2 \leq b|\xi|^2.$$

Given numbers positive numbers ν, κ and t_0, let $\mathcal{G}(\nu, \kappa, t_0)$ be the set of Riemannian metrics g on M such that every unit speed geodesic of g of length one experiences negative curvature $K_g \leq -\nu$ for at least a time t_0 and $K_g \leq \kappa$. Moreover, we require $|K_g| \leq 2$. The next lemma is the key for the construction.

LEMMA 5.11. *Given $\nu, t_0 > 0$, there exists $\kappa_0 > 0$ sufficiently small such that for all $\kappa \leq \kappa_0$ any metric $g \in \mathcal{G}(\nu, \kappa, t_0)$ is Anosov.*

PROOF. Let $Q_{(x,v)}$ be the quadratic form (5.1.1) used in the proof of Theorem 1.6. Consider a new quadratic form defined by

$$Q'_{(x,v)}(\xi) = \int_0^1 Q_{\phi_t(x,v)}(d\phi_t(\xi))dt.$$

EXERCISE 5.12. Show that the Lie derivative of Q' along X is given by

$$L_X Q'_{(x,v)}(\xi) = \int_0^1 (\dot{y}^2 - Ky^2)dt$$

where y is the solution of the Jacobi equation $\ddot{y} + Ky = 0$ with $y(0) = y$ and $\dot{y}(0) = z$. Conclude that Q' is non-degenerate on $\ker \alpha$.

From now on we might as well assume that ξ is such that $a = 0$. Therefore it suffices to show that for all κ sufficiently small

$$\int_0^1 (\dot{y}^2 - Ky^2)dt > 0 \tag{5.2.1}$$

provided that $\xi \neq 0$. Without loss of generality suppose that the geodesic starting at (x, v) experiences negative curvature $\leq -\nu$ in $[0, t_0]$. Clearly, we have:

$$\int_0^1 (\dot{y}^2 - Ky^2)dt \geq \int_0^{t_0} (\dot{y}^2 + \nu y^2)dt - \kappa \int_{t_0}^1 y^2 dt.$$

Since $|d\phi_t(\xi)|^2 = \dot{y}^2(t) + y^2(t)$, on account of Exercise 5.10, there are positive constants a and b such that for all $t \in [0, 1]$ and ξ

$$a|\xi|^2 \leq \dot{y}^2(t) + y^2(t) \leq b|\xi|^2.$$

Therefore

$$\int_0^1 (\dot{y}^2 - Ky^2)dt \geq a \min\{1, \nu\} t_0 |\xi|^2 - \kappa b(1 - t_0)|\xi|^2$$

and (5.2.1) follows by choosing

$$\kappa < \frac{a \min\{1, \nu\} t_0}{(1 - t_0)b}.$$

□

Here is the main idea of the construction, for the details we refer the reader to [**DP03, DP04**]. Consider a sphere S_n in $\mathbb{R}^3$ with radius 2^n for n large so that it is nearly flat. Drill in sufficiently many holes in it, so that every unit speed geodesic of length one meets the boundary of a hole at an angle at least $\phi > 0$. Consider a concentric copy of S_n, say S_n', which is shrunk by $1 - 2^{-n}$. Connect the boundaries of matching holes in S_n and S_n' by dispersing tubes. Note that this requires a modification of the metric on the spheres near the boundaries of the holes.

There are three types of geodesics in the resulting surface M. The closed geodesics Γ around the waist of each tube, the geodesics that are asymptotic to these closed geodesics and the geodesics that enter and exit dispersing tubes at an angle $\geq \phi$.

Now, the point is that every unit speed geodesic of unit length experiences negative curvature $K \leq -\nu$ for at least a time t_0 because it enters a tube at an angle $\geq \phi$. Also the curvature of M is $\leq \kappa$ where κ is the maximum of the curvature of the surface. If we take n very large, then κ becomes as small as we wish, but ν and t_0 stay fixed since the dispersing tubes stay roughly of the same size and the entry angle is always at least ϕ.

We can now apply Lemma 5.11 to conclude that for n large the geodesic flow of M is Anosov.

CHAPTER 6

Livsic Theory

In this chapter we give an outline of Livsic theory. Principally this consists of three separate results, the *Livsic Periodic Data Theorem*, the *Livsic Cocycle Regularity Theorem* and the *Measurable Livsic Theorem*. We will essentially give two versions of each; the first is the *commutative* case, and it is this version that we will use throughout the book. The second version is the *noncommutative* case; we will need this only for the last chapter in the book, and so the reader may omit these until later (specifically, the somewhat technical Section 6.2 can be omitted until the final chapter). This chapter is in five parts. In the last section of the chapter we discuss flow cohomology, and outline how we will use this to prove Theorem 1.9.

6.1. The Commutative Livsic Periodic Data Theorem

We need the following result in order to get started on proving the first version we present of the Livsic Periodic Data Theorem. It is a simple consequence of the Anosov Closing Lemma (see Theorem 6.4) and we will give a proof at the end of Section 6.2 (see Proposition 6.10).

PROPOSITION 6.1. *Let $\phi_t : N \to N$ be a transitive Anosov flow on the closed manifold N, and $f : N \to \mathbb{R}$ an α-Hölder continuous function. Then there exists $\varepsilon > 0$, $K_0 > 0$ and $T_0 > 0$ such that if $d(\phi_T x, x) < \varepsilon$ for some $T > T_0$ then there exists a closed orbit Γ with period $T + \tau$ for some $\tau \geq 0$ such that*

$$\left| \int_0^{T+\tau} f(\phi_t p)dt - \int_0^T f(\phi_t x)dt \right| \leq K_0 d(\phi_T x, x)^\alpha,$$

where p is some point in Γ.

DEFINITION 6.2. Let N be a closed manifold and ϕ_t a flow on N, and $f : N \to \mathbb{R}$ a continuous function. We say that f satisfies the *periodic orbit obstruction condition* if for every closed orbit Γ we have

$$\int_0^\tau f(\phi_t x_0)dt = 0, \tag{6.1.1}$$

where τ is the period of Γ and x_0 is any point in Γ.

Here then is the *Livsic Periodic Data Theorem*.

THEOREM 6.3. *Let N be a closed manifold and ϕ_t a transitive Anosov flow on N, and $f : N \to \mathbb{R}$ an α-Hölder continuous function satisfying the periodic orbit obstruction condition. Then there exists an α-Hölder continuous function $u : N \to \mathbb{R}$ such that u is differentiable in the flow direction and $X(u) = f$, where X is the infinitesimal generator of ϕ_t.*

Note that u being differentiable in the flow direction and $X(u) = f$ is equivalent to requiring that

$$u(\phi_t x) - u(x) = \int_0^t f(\phi_s x)ds, \tag{6.1.2}$$

by the fundamental theorem of calculus.

PROOF. Let Γ_0 be a dense orbit of ϕ_t and $x_0 \in \Gamma_0$. First define $u : \Gamma_0 \to \mathbb{R}$ by

$$u(\phi_t x_0) = \int_0^t f(\phi_s x_0)ds.$$

Then we claim that u is α-Hölder continuous on Γ_0.

Take $x, y \in \Gamma_0$ such that $d(x, y) < \varepsilon$, where ε is as in the statement of the Proposition 6.1. Suppose $x = \phi_r x_0$ and $y = \phi_s x_0$ where without loss of generality $s \geq r$ and $s - r > T_0$. Then $d(\phi_{s-r}x, x) < \epsilon$. Hence Proposition 6.1 implies the existence of a closed orbit Γ of ϕ_t such that

$$\left| \int_0^{s-r} f(\phi_t z_0)dt - \int_0^{s-r} f(\phi_t x)dt \right| \leq K_0 d(\phi_{s-r}x, x)^\alpha = K_0 d(y, x)^\alpha,$$

where $z_0 \in \Gamma$ is any point.

But since f satisfies the periodic orbit obstruction condition, the first term is zero, whence we obtain

$$|u(y) - u(x)| = \left| \int_0^{s-r} f(\phi_t x)dt \right| \leq K_0 d(y, x)^\alpha. \tag{6.1.3}$$

Thus u is α-Hölder continuous on Γ_0 as claimed. Moreover it is clear that (6.1.2) holds on Γ_0.

It is now easy to complete the proof. Since u is α-Hölder on Γ_0, in particular it is uniformly continuous, and thus there exists a unique extension of u to a function $u : \overline{\Gamma_0} = N \to \mathbb{R}$. We can pass to the limit in both (6.1.2) and (6.5.1) to conclude that u is α-Hölder continuous on all of N and is differentiable along the flow direction and satisfies $X(u) = f$. □

6.2. The Noncommutative Livsic Periodic Data Theorem

Our treatment of the Noncommutative Livsic Periodic Data Theorem is based on [**dlLW07**, Lemma 2.4]. We begin with the following result, known as the *Anosov Closing Lemma*; the precise statement given below is that of [**dlLW07**, Lemma 2.4]; for the proof we refer the reader to [**KH95**, Corollary 18.1.8].

THEOREM 6.4. *Let $\phi_t : N \to N$ be a transitive Anosov flow on a closed manifold N. Then there exists $\varepsilon > 0$, $K > 0$ and $T_0 > 0$ such that if for some $T > T_0$,*

$$d(\phi_T x, x) < \varepsilon,$$

then there exists a unique periodic point $p \in N$ with period $T + \tau$ such that

$$\max \{d(x, p), d(\phi_T x, p), |\tau|\} \leq K\varepsilon$$

and

$$W^s_{\text{loc}}(p) \cap W^u_{\text{loc}}(x) \neq \emptyset.$$

In fact, this unique point p in addition satisfies

$$\max \{d(x, p), d(\phi_T x, p), |\tau|\} \leq K d(\phi_T x, x),$$

and there exists a unique point $z \in N$ such that

$$W^s_{\text{loc}}(p) \cap W^u_{\text{loc}}(x) = \{z\}.$$

At the end of this section we shall show how to deduce Proposition 6.1 from Theorem 6.4.

DEFINITION 6.5. Let G be a compact Lie group. A *G-valued cocycle ϕ_t* is a map $C : N \times \mathbb{R} \to G$ that satisfies

$$C(x, t + s) = C(\phi_t x, s)C(x, t)$$

for all $x \in N$ and $s, t \in \mathbb{R}$, where the product denotes multiplication in G.

Let $\mathfrak{g}$ denote the Lie algebra of G. If C is smooth then we can consider its infinitesimal generator $A : N \to \mathfrak{g}$ defined by

$$A(x) := \frac{d}{dt}\Big|_{t=0} C(x, t).$$

In fact, we can recover C from A:

LEMMA 6.6. *C is the unique solution to the ODE*

$$\frac{d}{dt}C(x, t) = dR_{C(x,t)}(A(\phi_t x)) \tag{6.2.1}$$

subject to the initial condition

$$C(x, 0) = e, \tag{6.2.2}$$

where $R_g : G \to G$ is right translation by $g \in G$, and e is the identity element of G.

PROOF. We have

$$\begin{aligned} dR_{C(x,t)}(A(\phi_t x)) &= dR_{C(x,t)}\left(\frac{d}{ds}|_{s=0}C(\phi_t x, s)\right) \\ &= dR_{C(x,t)}\left(\frac{d}{ds}|_{s=0}C(x,t+s)\cdot C(x,t)^{-1}\right) \\ &= dR_{C(x,t)}\left(dR_{C(x,t)^{-1}}\left(\frac{d}{ds}|_{s=0}C(x,t+s)\right)\right) \\ &= \frac{d}{dt}C(x,t), \end{aligned}$$

and uniqueness follows from standard ODE theory. □

Thus more generally, if we are given a continuous function $A : N \to \mathfrak{g}$ we can define a cocycle C by the equations (6.2.1) and (6.2.2).

Here is the analogue of Definition 6.2.

DEFINITION 6.7. Let $\phi_t : N \to N$ be a flow on a closed manifold N, and G a compact Lie group. Let C be a G-valued cocycle. We say that C satisfies the *periodic orbit obstruction condition* if for all $x \in N$ and $T \in \mathbb{R}$,

$$\phi_T x = x \quad \Rightarrow \quad C(x,T) = e.$$

We now state our main theorem of this chapter, the *Livsic Periodic Data Theorem*. We will use this theorem in the final chapter of these notes (see Theorem 13.9).

THEOREM 6.8. *Let $\phi_t : N \to N$ be a transitive Anosov flow on a closed manifold N, and G a compact Lie group with Lie algebra $\mathfrak{g}$. Let $A : N \to \mathfrak{g}$ be a Hölder continuous function and C the corresponding G-valued cocycle defined by equations* (6.2.1) *and* (6.2.2)*. Suppose in addition that C satisfies the periodic orbit obstruction condition. Then there exists an Hölder continuous function $u : N \to G$ such that for all $x \in N$ and $t \in \mathbb{R}$,*

$$C(x,t) = u(\phi_t x)u(x)^{-1}. \tag{6.2.3}$$

Before starting the proof, we will need the following two remarks.

REMARK 6.9. If G is a compact Lie group then G admits a bi-invariant metric d_G, that is, a metric $d_G : G \times G \to \mathbb{R}$ such that for all $g, h, k \in G$ we have

$$d_G(g\cdot h, g\cdot k) = d_G(h,k) = d_G(h\cdot g, k\cdot g).$$

Indeed, such a metric is obtained as the geodesic metric of a bi-invariant Riemannian metric $\langle\cdot,\cdot\rangle$ on G as follows: select a left-invariant metric $(\cdot,\cdot)$ and a volume form $\omega = \sigma^1 \wedge \cdots \wedge \sigma^{\dim G}$ where the σ^i are left-invariant 1-forms. Then define

$$\langle v, w\rangle := \frac{1}{\int_G \omega(x)}\int_G (dR_x(v), dR_x(w))\omega(x).$$

Then $\langle\cdot,\cdot\rangle$ is bi-invariant. See [**GHL04**, Section 2.46] for more information.

We will also need the following elementary inequality during the proof of Theorem 6.8. By Lemma 6.6, if C and A are as above, we have

$$\frac{d}{dt}C(x,t) = dR_{C(x,t)}(A(\phi_t x)),$$

we thus can estimate

$$\begin{aligned} d_G(C(x,t),e) &\le \left|\int_0^t dR_{C(x,s)}(A(\phi_s x))ds\right| \\ &\le \int_0^t |A(\phi_s x)|\,ds. \end{aligned} \tag{6.2.4}$$

Without further ado, we begin the proof of Theorem 6.8

PROOF. *(of Theorem 6.8)*

In this proof we will let K denote an arbitrary constant, which may change from line to line. Let α be the Hölder exponent of A. Fix a point $x_0 \in N$ such that the orbit Γ of x_0 is dense in N. We begin by defining our function u on Γ. Choose a point $u(x_0) \in G$ and then set

$$u(\phi_t x_0) := C(x_0, t) \cdot u(x_0).$$

We now show that u is α-Hölder continuous on Γ. Apply Theorem 6.4 with $x := \phi_t x_0$ to deduce the existence of $\epsilon, K, T_0 > 0$ such that if $T > T_0$ and

$$d(\phi_{t+T} x_0, \phi_t x_0) < \epsilon$$

then there exists a unique periodic point p with period $T + \tau$ where $\tau = \tau(T)$ is bounded for all T, and

$$d(\phi_{t+T} x_0, p) \leq K d(\phi_{t+T} x_0, \phi_t x_0), \tag{6.2.5}$$

together with a point $z \in W^s(p) \cap W^u(\phi_t x_0)$.

Now note that

$$\begin{aligned} d_G(u(\phi_{t+T} x_0), u(\phi_t x_0)) &= d_G(C(x_0, t+T) \cdot u(x_0), C(x_0, t) \cdot u(x_0)) \\ &= d_G(C(x_0, t+T), C(x_0, t)) \\ &= d_G(C(\phi_t x_0, T) \cdot C(x_0, t), C(x_0, t)) \\ &= d_G(C(\phi_t x_0, T), e), \end{aligned}$$

where we repeatedly used bi-invariance and the cocycle property. Then by the triangle inequality and bi-invariance, we have

$$\begin{aligned} d_G(C(\phi_t x_0, T), e) &\leq d_G(C(\phi_t x_0, T), C(z, T)) + d_G(C(z, T), e) \\ &= d_G(C(\phi_t x_0, T) \cdot C(z, T)^{-1}, e) + d_G(C(z, T), e), \\ &\leq d_G(C(\phi_t x_0, T) \cdot C(z, T)^{-1}, e) + d_G(C(z, T), C(p, T)) \\ &\quad + d_G(C(p, T), e) \\ &= d_G(C(\phi_t x_0, T) \cdot C(z, T)^{-1}, e) + d_G\left(C(p, T)^{-1} \cdot C(z, T), e\right), \\ &\quad + d_G(C(p, T), e), \end{aligned}$$

and thus

$$\begin{aligned} d_G(u(\phi_{t+T} x_0), u(\phi_t x_0)) &\leq \underbrace{d_G(C(\phi_t x_0, T) \cdot C(z, T)^{-1}, e)}_{\text{(I)}} \\ &\quad + \underbrace{d_G(C(p, T)^{-1} \cdot C(z, T), e)}_{\text{(II)}} + \underbrace{d_G(C(p, T), e)}_{\text{(III)}}. \end{aligned} \tag{6.2.6}$$

We therefore need to estimate (I) , (II) and (III). More specifically, we will show that each one can be bounded by a term of the form $K d(\phi_{t+T} x_0, \phi_t x_0)^\alpha$.

<u>**Estimate of (I):**</u> Define

$$a(s) := C(\phi_{t+T} x_0, -s)^{-1} \cdot C(\phi_T z, -s).$$

Then by the chain rule,

$$\frac{d}{ds} a(s) = dL_{C(\phi_{t+T} x_0, -s)^{-1}} \circ dR_{C(\phi_T z, -s)}(A(\phi_{t+T-s} x_0) - A(\phi_{T-s} z)),$$

where $L_g : G \to G$ is left-translation by $g \in G$. Since dL_g and dR_g are d_G-isometries for all $g \in G$, we have

$$\begin{aligned} \left| \frac{d}{ds} a(s) \right| &= |A(\phi_{t+T-s} x_0) - A(\phi_{T-s} z)| \\ &\leq K d(\phi_{t+T-s} x_0, \phi_{T-s} z)^\alpha \\ &\leq K e^{-\lambda \alpha s} d(\phi_{t+T} x_0, \phi_T z)^\alpha \\ &\leq K e^{-\lambda \alpha s} d(\phi_{t+T} x_0, p)^\alpha \\ &\leq K e^{-\lambda \alpha s} d(\phi_{t+T} x_0, \phi_t x_0)^\alpha \end{aligned}$$

for some (varying) constant $K > 0$, where in the first inequality we used the fact A is α-Hölder continuous, the second and third used the fact that $\phi_T z \in W^u_{\mathrm{loc}}(\phi_{t+T} x_0)$, and the last used (6.2.5). Thus

$$d_G(a(s), e) \leq \frac{K}{\lambda\alpha} d(\phi_{t+T} x_0, \phi_t x_0)^\alpha, \tag{6.2.7}$$

and note that (6.2.7) is *uniform* in s.

Now take $s = T$ to obtain

$$\begin{aligned} a(T) &= C(\phi_{t+T} x_0, -T)^{-1} \cdot C(\phi_T z, -T) \\ &= \left\{C(\phi_t x_0, T - T) \cdot C(\phi_t x_0, T)^{-1}\right\}^{-1} \cdot \left\{C(z, T - T) \cdot C(z, T)^{-1}\right\}^{-1} \\ &= C(\phi_t x_0, T) \cdot C(z, T)^{-1}, \end{aligned}$$

and combining this with (6.2.7) we obtain

$$\underbrace{d_G(C(\phi_t x_0, T) \cdot C(z, T)^{-1}, e)}_{\text{(I)}} \leq K d(\phi_{t+T} x_0, \phi_t x_0)^\alpha,$$

which gives us an estimate for (I).

Estimate of (II): This is very similar to the estimate of (I) above. Define

$$b(s) := C(p, s)^{-1} \cdot C(z, s).$$

Then

$$\frac{d}{ds} b(s) = dL_{C(p,s)^{-1}} \circ dR_{C(z,s)} \left[A(\phi_s z) - A(\phi_s p)\right],$$

and as before we have

$$\begin{aligned} \left|\frac{d}{ds} b(s)\right| &= |A(\phi_s z) - A(\phi_s p)| \\ &\leq K d(\phi_s z, \phi_s p)^\alpha \\ &\leq K e^{-\lambda\alpha s} d(z, p)^\alpha \\ &\leq K e^{-\lambda\alpha s} d(\phi_{t+T} x_0, \phi_t x_0)^\alpha, \end{aligned}$$

where in the first inequality we used the fact A is α-Hölder continuous, the second used the fact that $z \in W^s(p)$, and the last used (6.2.5). Thus

$$d_G(b(s), e) \leq \frac{K}{\lambda\alpha} d(\phi_{t+T} x_0, \phi_t x_0)^\alpha,$$

again, uniform in s. Then as before take $s = T$ and note that $b(T) = C(p, T)^{-1} \cdot C(z, T)$, we have

$$\begin{aligned} \underbrace{d_G(C(p, T)^{-1} \cdot C(z, T), e)}_{\text{(II)}} &= d_G(b(T), e) \\ &\leq \frac{K}{\lambda\alpha} d(\phi_{t+T} x_0, \phi_t x_0)^\alpha, \end{aligned}$$

which finishes the estimate of (II).

Estimate of (III): To bound (III) we finally use the hypotheses that C satisfies the periodic orbit obstruction condition; since p is periodic with period $T + \tau$ we have

$$C(p, T + \tau) = e,$$

and thus

$$C(p, T) = C(p, -\tau)$$

by the cocycle property. Then by (6.2.4), we have

$$\begin{aligned} d_G(C(p, -\tau), e) &\leq \int_{-\tau}^{0} |A(\phi_s x_0)|\, ds. \\ &\leq K d(\phi_{t+T} x_0, \phi_t x_0)^\alpha, \end{aligned}$$

since $|\tau|$ is bounded and A is α-Hölder continuous. Thus

$$\underbrace{d_G(C(p,T),e)}_{\text{(III)}} \leq Kd(\phi_{t+T}x_0,\phi_t x_0)^\alpha,$$

which completes the estimate for (III).

Putting these estimates together into (6.2.6) then gives

$$d_G(u(\phi_{t+T}x_0),u(\phi_t x_0)) \leq \text{(I)} + \text{(II)} + \text{(III)} \leq Kd(\phi_{t+T}x_0,\phi_t x_0)^\alpha, \tag{6.2.8}$$

and thus u is Hölder continuous on Γ as claimed.

As in the proof of Theorem 6.3, it is now easy to complete the proof: since u is Hölder continuous it is in particular uniformly continuous, and thus admits a unique extension to a map $u : \bar{\Gamma} = N \to \mathfrak{g}$. Moreover passing to the limit in (6.2.3) and (6.2.8) then shows that u satisfies the conditions of the theorem and is α-Hölder continuous, and the proof is complete. □

We conclude this section by showing how to deduce Proposition 6.1 from Theorem 6.4. For the convenience of the reader we restate Proposition 6.1 as Proposition 6.10 below.

PROPOSITION 6.10. *Let $\phi_t : N \to N$ be a transitive Anosov flow on the closed manifold N, and $f : N \to \mathbb{R}$ an α-Hölder continuous function. Then there exists $\epsilon > 0$, $K_0 > 0$ and $T_0 > 0$ such that if $d(\phi_T x, x) < \epsilon$ for some $T > T_0$ then there exists a closed orbit Γ with period $T+\tau$ for some $\tau \geq 0$ such that*

$$\left| \int_0^{T+\tau} f(\phi_t p)dt - \int_0^T f(\phi_t x)dt \right| \leq K_0 d(\phi_T x, x)^\alpha,$$

where p is some point in Γ.

PROOF. By Theorem 6.4, we can find $\epsilon, K, T_0 > 0$ such that if $T > T_0$ and $d(\phi_T x, x) < \epsilon$ then there exists a unique periodic point $p \in N$ with period $T+\tau$ such that

$$\max\{d(x,p), d(\phi_T x, p), |\tau|\} \leq K \min\{\epsilon, d(\phi_T x, x)\}$$

and a unique point $z \in W^s_{\text{loc}}(p) \cap W^u_{\text{loc}}(x)$. Now we estimate

$$\begin{aligned} \left| \int_0^{T+\tau} f(\phi_t p)dt - \int_0^T f(\phi_t x)dt \right| \quad &\leq \quad \int_0^\tau |f(\phi_t p)|\, dt + \int_0^T |f(\phi_t p) - f(\phi_t z)|\, dt \\ &\quad + \int_0^T |f(\phi_t z) - f(\phi_t x)|\, dt. \end{aligned}$$

Arguments very similar to the proof of Theorem 6.8 above show that we can bound each separate term by a term of the form $Kd(\phi_T x, x)^\alpha$, and thus this completes the proof. □

6.3. The Livsic Cocycle Regularity Theorems

In this subsection and the next, we shall state two sets of results, both addressing the issue of regularity. The first deals with the following situation. Suppose we assume that f in Theorem 6.3 or C in Theorem 6.8 possesses higher regularity than just Hölder continuity. Does this imply that u has more regularity? Here is the precise answer. We call these two results the *Livsic Cocycle Regularity Theorems*.

THEOREM 6.11. *Let N be a closed manifold and ϕ_t a transitive Anosov flow on N with infinitesimal generator X, and $f : N \to \mathbb{R}$ a C^k function, for $k \in \mathbb{N} \cup \{\infty\}$ satisfying the periodic orbit obstruction condition. Then there exists a C^k function $u : N \to \mathbb{R}$ such that $X(u) = f$.*

THEOREM 6.12. *Let $\phi_t : N \to N$ be a transitive Anosov flow on a closed manifold N, and G a compact Lie group with Lie algebra $\mathfrak{g}$, $A : N \to \mathfrak{g}$ is a C^k function, for $k \in \mathbb{N} \cup \{\infty\}$ and C the corresponding G-valued cocycle defined by equations* (6.2.1) *and* (6.2.2)*. Suppose C satisfies the periodic orbit obstruction condition. Then there exists a C^k function $u : N \to G$ such that for every $x \in N$ and $t \in \mathbb{R}$, $C(x,t) = u(\phi_t x)u(x)^{-1}$.*

We will use the second of these two theorems in Chapter 13 (see Theorem 13.9).

REMARK 6.13. We will also state a different form of Theorem 6.11 which is the one we will use in several places throughout these notes. Its proof is but part of the proof of Theorem 6.11. It is important to note that this statement does not require either the transitivity of the flow or the periodic orbit obstruction condition.

THEOREM 6.14. *Suppose $\phi_t : N \to N$ is an Anosov flow on a closed manifold N with infinitesimal generator X. Suppose f is a C^k function for $k \in \mathbb{N} \cup \{\infty\}$, and u is a Hölder continuous function that is differentiable in the flow direction such that $X(u) = f$. Then there exists a C^k function $v : N \to \mathbb{R}$ such that $X(v) = f$.*

We will not attempt to prove Theorem 6.12 but we will sketch an outline of the proof of a special case of Theorem 6.11; see Corollary 6.16 below, in order to give an essence of how the proofs go. Theorem 6.11 is originally due to de La Llave, Marco and Moriyon (see [**dlLMM86**, Theorem 2.1]), although Livsic considered the C^1 case for abelian Lie groups in [**Liv71, Liv72**]. A different proof was given by Hurder and Katok in [**HK90**, Theorem 2.2]. The proof was then further simplified by Journé in [**Jou86**] (see Theorem 6.17, below). Theorem 6.12 is due to Niţică and Török in [**NT96, NT98**].

To begin our proof of a special case of Theorem 6.11, we first prove that u has the same regularity as f in the *stable* and *unstable* directions.

PROPOSITION 6.15. *Let N be a closed manifold and ϕ_t a transitive Anosov flow on N, and $f : N \to \mathbb{R}$ a C^k function, for $k \in \mathbb{N} \cup \{\infty\}$ satisfying the periodic orbit obstruction condition. Then there exists a Lipschitz continuous function $u : N \to \mathbb{R}$ that is differentiable in the flow direction and of class C^k along the stable and unstable manifolds such that $X(u) = f$.*

PROOF. By Theorem 6.3, we know that that u is Lipschitz continuous on all of N (since we may take $\alpha = 1$). Suppose now $y \in W^s(x)$. Using (6.1.2) we obtain

$$u(y) - u(x) = u(\phi_t y) - u(\phi_t x) + \int_0^t \{f(\phi_s x) - f(\phi_s y)\} ds.$$

Since $|f(\phi_s x) - f(\phi_s y)| \leq Ce^{-\lambda t} d(x, y)$ we conclude that the integral converges uniformly and thus we may let $t \to \infty$ to obtain

$$u(y) - u(x) = \int_0^\infty \{f(\phi_s x) - f(\phi_s y)\} ds.$$

Now take a curve $\gamma : (-\epsilon, \epsilon) \to W^s(x)$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v \in T_x W^s(x) = E^s(x)$. Then

$$\frac{u(\gamma(r)) - u(\gamma(0))}{r} = -\frac{1}{r} \int_0^\infty \{f(\phi_s(\gamma(r))) - f(\phi_s(\gamma(0)))\} ds,$$

and hence

$$\frac{d}{dr}\Big|_{r=0} u(\gamma(r)) = -\int_0^\infty df(d\phi_s(v)) ds.$$

Now since $v \in E^s(x)$ we have $d\phi_s(v) \to 0$ exponentially fast as $s \to \infty$, and hence the integral again converges uniformly. This shows that u is differentiable in the direction of the stable manifold $W^s(x)$. A similar argument shows that u is differentiable in the unstable direction. In fact, this shows that if f is C^k then u is C^k along the stable and unstable foliations. □

COROLLARY 6.16. *Let N be a 3-manifold, and ϕ^t and f satisfying the conditions of the Livsic theorem, where f is in addition C^1. Then u is C^1 on all of N.*

It is only for notational simplicity that we assume that N is 3-dimensional; the same proof clearly works for higher dimensions but is harder to write down.

PROOF. Let $\{X, Y^s, Y^u\}$ be a C^0 local frame on $U \subseteq N$ such that $\{X(x), Y^s(x), Y^u(x)\}$ is a basis of $T_x N$ for all $x \in U$ and such that for all $x \in U$,

$$E^s(x) = \operatorname{span}\{Y^s(x)\}$$

and

$$E^u(x) = \operatorname{span}\{Y^u(x)\}.$$

If (x^1, x^2, x^3) are local coordinates on U then for $i = 1, 2, 3$ we can find continuous functions a^i, b^i, c^i such that

$$\frac{\partial}{\partial x^i} = a^i X + b^i Y^s + c^i Y^u.$$

Then

$$\frac{\partial u}{\partial x^i} = a^i X(u) + b^i Y^s(u) + c^i Y^u(x),$$

which is continuous by Proposition 6.15. Hence u is C^1 as claimed. □

In fact, the following theorem, due to Journé [**Jou86**] allows us to deduce the full Theorem 6.11 from Proposition 6.15.

THEOREM 6.17. *Let N be a smooth manifold and $\mathcal{F}^s$ and $\mathcal{F}^u$ two Hölder transverse foliations with uniformly smooth leaves. Then if $u : N \to \mathbb{R}$ is uniformly C^k along the leaves of the two foliations then u is C^k on all of N.*

Finally we conclude this section by stating an enhanced version of Theorem 6.11; essentially this is a version 'with parameters'. Its proof can be found in [**dlLMM86**]. We will use this result once, in the proof of Theorem 12.7 in Chapter 12.

THEOREM 6.18. *Let N be a closed manifold, and $\{X_s\}$ a smooth family of vector fields (that is, $s \mapsto X_s$ is smooth) generating transitive Anosov flows $\{\phi_t^s\}$. If $\{f_s\}$ is a smooth family of functions (that is, $s \mapsto f_s$ is smooth) such that for every closed orbit Γ_s of the flow ϕ_t^s of X_s we have*

$$\int_0^{T(\Gamma_s)} f_s(\phi_t^s x_0^s)dt = 0,$$

where $T(\Gamma_s)$ is the period of Γ_s and x_0^s is some point in Γ_s, then the smooth functions u_s such that

$$X_s(u_s) = f_s,$$

(whose existence is guaranteed Theorem 6.11) actually form a smooth family of functions, that is, $s \mapsto u_s$ is also smooth.

6.4. The Measurable Livsic Theorems

We will now move on to the second set of regularity results. Essentially they both say the same thing: if instead of assuming (in either Theorem 6.3 or Theorem 6.8) that the periodic orbit obstruction condition is satisfied, but instead we assume the existence of a *measurable* function w satisfying the conclusions of the theorem almost everywhere then both theorems still hold, that is, there exists an α-Hölder continuous function u satisfying the conclusions of the theorem. Together we call them the *Measurable Livsic Theorems*. We will use Theorem 6.19 in Chapter 11 (see Theorem 11.8).

THEOREM 6.19. *Let N be a closed manifold and ϕ_t a transitive Anosov flow on N, and $f : N \to \mathbb{R}$ an α-Hölder continuous function. Suppose there exists a measurable function w that is differentiable in the flow direction and satisfies $X(w) = f$ almost everywhere, where X is the infinitesimal generator of ϕ_t. Then there exists an α-Hölder continuous function $u : N \to \mathbb{R}$ such that u is differentiable in the flow direction and $X(u) = f$.*

THEOREM 6.20. *Let $\phi_t : N \to N$ be a transitive Anosov flow on a closed manifold N, and G a compact Lie group with Lie algebra $\mathfrak{g}$. Let $A : N \to \mathfrak{g}$ be a Hölder continuous function and C the corresponding G-valued cocycle defined by equations* (6.2.1) *and* (6.2.2). *Suppose there exists a measurable function $w : N \to G$ such that for almost every $x \in N$ and $t \in \mathbb{R}$, $C(x,t) = w(\phi_t x)w(x)^{-1}$. Then there exists an Hölder continuous function $u : N \to G$ such that for every $x \in N$ and $t \in \mathbb{R}$, $C(x,t) = u(\phi_t x)u(x)^{-1}$.*

We shall not try to prove either theorem; the first is due to Livsic and Sinai (see [**LS72**]), and a version of the second is due to Parry and Pollicott (see [**PP97**]). A generalization of Theorem 6.20 is given in [**Wal00**, Section 6], which also ties together this result with Theorem 6.12, as well as giving further historical comments on all of these results.

6.5. The cohomological equation

We now return to Theorem 1.9. That is, let (M, g) be a closed surface whose geodesic flow ϕ_t is Anosov, and let $h \in C^\infty(M)$ and $\theta \in \Omega^1(M)$. Suppose $I[h+\theta] = 0$. Define $f : SM \to \mathbb{R}$ by

$$f(x, v) = h(x) + \theta_x(v).$$

Then since $I[h+\theta] = 0$, the conditions of the Theorem 6.11 are satisfied, and we conclude that there exists a smooth function $u : SM \to \mathbb{R}$ such that $X(u) = f$, where X is the geodesic vector field (note that Theorem 3.12 and Corollary 3.14 gives transitivity of ϕ_t).

The PDE

$$X(u) = f$$

is called the *cohomological equation*.

DEFINITION 6.21. We can do cohomology with functions: if $f, g : SM \to \mathbb{R}$ are smooth functions we say that f is *flow cohomologous* to g if there exists a smooth $u : SM \to \mathbb{R}$ such that $X(u) = f - g$ (the reason for this will become clear in Chapter 9 - see Remark 9.4). In particular we say $f \in C^\infty(SM, \mathbb{R})$ is a *coboundary* if $f = X(u)$ for some $u \in C^\infty(SM, \mathbb{R})$.

LEMMA 6.22. *Suppose we could show that the function $u : SM \to \mathbb{R}$ we obtain such that*

$$X(u) = h(x) + \theta_x(x, v) \tag{6.5.1}$$

*is **independent** of v. Then Theorem 1.9 is true.*

PROOF. If $\gamma_{(x,v)}(t) := \pi \circ \phi_t(x, v)$ is the unique geodesic on M adapted to (x, v) then if $u(x, v) = g(x)$ for some smooth function $g : M \to \mathbb{R}$ then we would have

$$\begin{aligned} X(u) &= \frac{d}{dt}\Big|_{t=0} u(\phi_t(x, v)) \\ &= \frac{d}{dt}\Big|_{t=0} g(\gamma_{(x,v)}(t)) \\ &= dg_x(v), \end{aligned}$$

and thus

$$dg_x(v) = h(x) + \theta_x(v). \tag{6.5.2}$$

Then by considering $(x, -v)$ we obtain $h(x) = 0$, and thus $h \equiv 0$ as x was arbitrary, and then (6.5.2) reads

$$dg = \theta,$$

so θ is indeed exact. □

It however will not be easy to show that u is independent of v; showing this will take all of Chapters 8 and 9. Before doing so however we will investigate in the next chapter an enlarged class of flows.

CHAPTER 7

λ-geodesic flows

In this chapter we enlarge the class of flows we study; instead of just geodesic flows we study λ-*geodesic flows*, which we show include all magnetic and thermostat flows. Finally we formulate a version of Theorem 1.9 for λ-geodesic flows.

7.1. Enlarging the class of flows

DEFINITION 7.1. Let (M, g) be a closed surface, and let $\frac{D}{dt}$ denote the covariant derivative associated to the Levi-Civita connection ∇. Let $\mathtt{i}$ denote the almost complex structure on M induced by g (see (4.1.2)). Let $\lambda \in C^\infty(SM, \mathbb{R})$ be an arbitrary smooth function. The λ-*geodesic equation* is

$$\frac{D\dot\gamma}{dt} = \lambda(\gamma, \dot\gamma)\mathtt{i}\dot\gamma \tag{7.1.1}$$

(note that the standard geodesic equation (1.1.1) is the special case $\lambda = 0$). We call curves that satisfy (7.1.1) λ-*geodesics*.

EXERCISE 7.2. Verify that as in the standard geodesic case, given $(x, v) \in SM$ there exists a unique λ-geodesic $\gamma_{(x,v)}$ adapted to (x, v), and that we can define a flow $\phi_t : SM \to SM$ by

$$\phi_t(x, v) = (\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t)). \tag{7.1.2}$$

DEFINITION 7.3. Let (M, g) be a closed surface and $\lambda \in C^\infty(SM, \mathbb{R})$. The λ-*geodesic flow* ϕ_t is the flow (7.1.2). To distinguish from the standard case of the geodesic flow, we now let F denote the infinitesimal generator of ϕ_t.

Recall now our moving frame $\{X, H, V\}$ with coframe $\{\alpha, \beta, \psi\}$ from Chapter 4 (see Summary 4.12). The infinitesimal generator can be expressed as a linear combination of the vector fields X, H, V, that is, there exist smooth functions $a, y, z : SM \to \mathbb{R}$ such that

$$F = aX + yH + zV.$$

Namely, $a = \alpha(F)$, $y = \beta(F)$ and $z = \psi(V)$. In fact, the following lemma shows that these functions a, y, z take a particularly simple form.

LEMMA 7.4. *The infinitesimal generator satisfies $F = X + \lambda V$.*

PROOF. Recalling the definition of the moving coframe $\{\alpha, \beta, \psi\}$ we have for $\gamma = \gamma_{(x,v)}$,

$$\begin{aligned} \psi(F(x, v)) &= \langle \mathcal{K}(F(x, v)), \mathtt{i}\dot\gamma\rangle \\ &= \left\langle \frac{D\dot\gamma}{dt}, \mathtt{i}\dot\gamma \right\rangle \\ &= \langle \lambda(\gamma, \dot\gamma)\mathtt{i}\dot\gamma, \mathtt{i}\dot\gamma\rangle \\ &= \lambda(\gamma, \dot\gamma). \end{aligned}$$

Similarly

$$\begin{aligned} \alpha(F(x, v)) &= \langle d\pi(F(x, v)), v\rangle \\ &= \langle \dot\gamma, \dot\gamma\rangle \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \beta(F(x, v)) &= \langle d\pi(F(x, v)), \mathtt{i}v\rangle \\ &= \langle \dot\gamma, \mathtt{i}\dot\gamma\rangle \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} F &= \alpha(F)X + \beta(F)H + \psi(F)V \\ &= X + \lambda V. \end{aligned}$$

□

In order to show that enlarging the class of flows we consider is a worthwhile thing to do, in the next two sections we consider two special cases of this new type of flow.

7.2. Magnetic Flows

DEFINITION 7.5. Let σ be a closed 2-form on M. Since M is an orientable surface, there exists a smooth function $f : M \to \mathbb{R}$ such that

$$\sigma = f\Omega_a, \tag{7.2.1}$$

where Ω_a is the area form from Definition 4.11.

Consider now the 2-form $\omega_\sigma \in \Omega^2(TM)$ defined by

$$\omega_\sigma := -d\alpha + \pi^*\sigma$$

where $\pi : TM \to M$ is the footpoint map.

EXERCISE 7.6. Prove that ω_σ is a symplectic form.

We call the symplectic manifold (TM, ω_σ) a *twisted tangent bundle*. Let $H : TM \to \mathbb{R}$ be the *energy Hamiltonian* from Lemma 1.3 defined by

$$H(x, v) = \frac{1}{2}|v|^2.$$

The *magnetic flow* or *twisted geodesic flow* associated to ω_σ is defined to be the flow of the unique vector field $\tilde{F}$ on TM defined by

$$i_{\tilde{F}}\omega_\sigma = dH,$$

i.e., it is the Hamiltonian flow of H with respect to the twisted symplectic form ω_σ. It is physically relevant, as it models the motion of a particle of unit mass and unit charge under the effect of a magnetic field (which is represented by the 2-form σ). See for instance [**AG90, Gin96**] for more information.

Set $\lambda := f \circ \pi : SM \to \mathbb{R}$, where f is as in (7.2.1). Let ϕ_t denote the λ-geodesic flow with infinitesimal generator F.

LEMMA 7.7. *$\tilde{F}|_{SM} = F$, that is, ϕ_t is the magnetic flow restricted to SM.*

PROOF. Note that λ extends to a function in all TM. Similarly, the vector fields X and V are also defined in all TM. We only need to show that on TM we have $i_{X+\lambda V}\omega_\sigma = dH$. Since $\pi^*\sigma$ is annihilated on vertical vectors we have

$$\begin{aligned} i_{X+\lambda V}\omega_\sigma &= -i_X d\alpha - \lambda i_V d\alpha + i_X\pi^*\sigma \\ &= dH - \lambda i_V d\alpha + i_X\pi^*\sigma. \end{aligned}$$

Thus we only need to check that $i_V d\alpha = i_X\pi^*\Omega_a$. Take $(x, v) \in TM$ and $\xi \in T_{(x,v)}TM$ and compute

$$\begin{aligned} i_V d\alpha(\xi) &\overset{(*)}{=} \langle d\pi(\xi), \mathtt{i}v\rangle \\ &= \Omega_a(v, d\pi(\xi)) \\ &= i_X\pi^*\Omega_a(\xi), \end{aligned}$$

where $(*)$ used Lemma 4.3. □

We have shown that magnetic flows form a subset of λ-geodesic flows, namely, magnetic flows restricted to SM are precisely the λ-geodesic flows where λ is of the form $\lambda = f \circ \pi$ for some $f \in C^\infty(M, \mathbb{R})$. In the next section we shall see that taking λ to be a 1-form allows us to model another physically relevant situation.

7.3. Thermostats

The geodesic flow is studied for several reasons. From a dynamical point of view, the geodesic flow is a ‘model’ of a conservative dynamical system, and thus is studied as a prototype to more realistic systems. Here is another, more geometrical point of view. Suppose $\bar{\nabla}$ is an arbitrary affine connection on M with associated covariant derivative $\frac{\bar{D}}{dt}$. Then a curve γ is a geodesic *with respect to* $\bar{\nabla}$ if $\frac{\bar{D}\dot{\gamma}}{dt} = 0$, and the corresponding flow $\bar{\phi}_t$ is the *geodesic flow arising from* $\bar{\nabla}$. From this point of view, the (standard) geodesic flow is the geodesic flow arising from the distinguished Levi-Civita connection, and thus is important in its own right.

We will now study *thermostats*, which are an extension of the above. From a dynamical point of view, these systems are studied as a model of a dissipative dynamical system. From the geometric point of view, these are the flows arising from metric connections that are not necessarily torsion-free.

Here is the definition.

DEFINITION 7.8. Let (M, g) be a closed surface. Suppose $E : M \to TM$ is a vector field on M. Let ∇ denote the Levi-Civita connection and $\frac{D}{dt}$ the associated covariant derivative. Consider the equation

$$\frac{D\dot{\gamma}}{dt} = E(\gamma) - \frac{\langle E(\gamma), \dot{\gamma}\rangle\, \dot{\gamma}}{|\dot{\gamma}|^2}. \tag{7.3.1}$$

The reason behind the strange looking term $\frac{\langle E(\gamma),\dot{\gamma}\rangle\dot{\gamma}}{|\dot{\gamma}|^2}$ is given in the next exercise: this is *‘Gauss’ least constraint principle’*; see [**Hoo86**].

EXERCISE 7.9. Show that if γ satisfies (7.3.1) then $H(\gamma(t), \dot{\gamma}(t))$ is constant, where H is the energy Hamiltonian from Lemma 1.3. Conclude that if ϕ_t is the flow defined by (7.3.1) then ϕ_t restricts to define a flow on SM.

We can write for $v \in S_xM$

$$E(x) = \langle E(x), v\rangle\, v + \langle E(x), \mathtt{i}v\rangle\, \mathtt{i}v, \tag{7.3.2}$$

and thus

$$\begin{aligned} \frac{D\dot{\gamma}}{dt} &= \langle E(\gamma), \dot{\gamma}\rangle\, \dot{\gamma} + \langle E(\gamma), \mathtt{i}\dot{\gamma}\rangle\, \mathtt{i}\dot{\gamma} - \langle E(\gamma), \dot{\gamma}\rangle\, \dot{\gamma} \\ &= \lambda(\gamma, \dot{\gamma})\mathtt{i}\dot{\gamma}, \end{aligned} \tag{7.3.3}$$

where

$$\lambda(x, v) = \langle E(x), \mathtt{i}v\rangle\,.$$

In other words, if

$$\theta := \hat{g}E$$

is the g-dual 1-form to E then

$$\lambda = V(\theta).$$

Conversely if λ is defined to be $V(\theta)$ for some 1-form θ and $E := \hat{g}^{-1}\theta$ then the thermostat flow of E is the λ-geodesic flow of λ. Thus thermostats are just another special case of λ-geodesic flows.

We now want to give the aforementioned geometric interpretation. First of all, however, we need to recall a more general version of the *Fundamental Theorem of Riemannian Geometry*.

THEOREM 7.10. *Let (M, g) be a Riemannian manifold and A an arbitrary antisymmetric 2-tensor. Then there exists a unique metric connection $\bar{\nabla}$ with torsion A.*

EXERCISE 7.11. Prove the theorem in the case where $\dim M = 2$ *(Hint: show there exists a unique 1-form θ such that*

$$A(X, Y) = \theta(Y)X - \theta(X)Y,$$

and try

$$\bar{\nabla}_X Y = \nabla_X Y - \langle X, Y\rangle\, E + \theta(Y)X$$

where $E = \hat{g}^{-1}\theta$ and ∇ is the Levi-Civita connection).

Let (M, g) be a closed surface and A an arbitrary antisymmetric 2-tensor. Let $\bar{\nabla}$ denote the connection given by Theorem 7.10. We now ask the question: what are the geodesics of $\bar{\nabla}$? If $\frac{\bar{D}}{dt}$ is the associated covariant derivative then

$$\frac{\bar{D}\dot{\gamma}}{dt} = 0 \quad \Leftrightarrow \quad \frac{D\dot{\gamma}}{dt} = \langle \dot{\gamma}, \dot{\gamma} \rangle \, E(\gamma) - \theta(\dot{\gamma})\dot{\gamma},$$

where $\frac{D}{dt}$ is the covariant derivative of the Levi-Civita connection and E and θ are as in Exercise 7.11. Since $\bar{\nabla}$ is metric, we may assume $|\dot{\gamma}| = 1$ for any geodesic γ, whence we obtain

$$\begin{aligned} \frac{D\dot{\gamma}}{dt} &= 1 \cdot E(\gamma) - \theta(\dot{\gamma})\dot{\gamma} \\ &= E(\gamma) - \langle E(\gamma), \dot{\gamma} \rangle \, \dot{\gamma} \\ &= \langle E(\gamma), \mathtt{i}\dot{\gamma} \rangle \, \mathtt{i}\dot{\gamma} \\ &= \lambda(\gamma, \dot{\gamma})\mathtt{i}\dot{\gamma}, \end{aligned}$$

where $\lambda = V(\theta)$. We have thus proved the following observation due to Wojtkowski and Przytycki [**WP08**]:

LEMMA 7.12. *The λ-geodesic flows where $\lambda = V(\theta)$ for $\theta \in \Omega^1(M)$ are precisely the geodesic flows arising from metric connections; namely if $\lambda = V(\theta)$ then the λ-geodesic flow is the geodesic flow of the unique metric connection $\bar{\nabla}$ with torsion $A(X, Y) := \theta(Y)X - \theta(X)Y$.*

We conclude this section with the following exercise.

EXERCISE 7.13. Suppose (M, g) is a closed surface with Gaussian curvature K, and $\phi_t : SM \to SM$ is the flow of a thermostat E. Let $\operatorname{div} E$ denote the divergence of E with respect to the area form Ω_a, that is, $\operatorname{div} E : M \to \mathbb{R}$ is uniquely defined by $L_E \Omega_a = \operatorname{div} E \, \Omega_a$.

Use Theorem 5.2 to show that if $K + \operatorname{div} E$ is negative, then ϕ_t is Anosov. (For this you will need to derive the Jacobi equations for thermostats as we have done for geodesic flows.)

7.4. The X-ray transform and λ-geodesic flows

This should have convinced the reader that this enlarged collection of flows is worth studying. We conclude this chapter by stating a generalization of Theorem 1.9 to the case of this new type of λ-geodesic flows. It will be proved in Chapter 9. First, an obvious definition.

DEFINITION 7.14. Let (M, g) be a closed surface and $\lambda \in C^\infty(SM, \mathbb{R})$. Let ϕ_t be the λ-geodesic flow on SM. We can define the *λ-geodesic X-ray transform I* in exactly the same way as the standard X-ray transform, only we now integrate over λ-geodesics. Namely, let $\mathscr{G}_\lambda(M, g)$ denote the set of closed λ-geodesics and define I as before in Definition 1.8 (on functions, 1-forms, symmetric 2-tensors etc), only replacing $\mathscr{G}(M, g)$ with $\mathscr{G}_\lambda(M, g)$.

Here is the generalization of Theorem 1.9 to λ-geodesic flows, due to Dairbekov and Paternain [**DP07**, Theorem B]. Earlier proofs of special cases of this result for some geodesic and magnetic flows (one of which being 1.9) using Fourier analysis exist; see Guillemin and Kazhdan [**GK80**] and Paternain [**Pat05**].

THEOREM 7.15. *Let (M, g) be a closed surface and $\lambda : SM \to \mathbb{R}$ any smooth function. Let ϕ_t be the λ-geodesic flow on SM. Suppose ϕ_t is Anosov, and that $h \in C^\infty(M, \mathbb{R})$ and $\theta \in \Omega^1(M)$ are such that*

$$I\,[h + \theta] = 0,$$

where I is the X-ray transform (integrating over λ-geodesics). Then $h \equiv 0$ and θ is exact.

CHAPTER 8

The Maslov cycle and the Riccati equation

This chapter is dedicated to proving Theorem 8.2 below. An important consequence of this, explained at the end of the chapter is the Riccati equations, Proposition 8.19. We use these in the next chapter to deduce Theorem 7.15.

8.1. The Maslov cycle

DEFINITION 8.1. Let (M, g) be a closed surface and ϕ_t an Anosov λ-geodesic flow on SM with infinitesimal generator F. Since ϕ_t is Anosov we have a splitting $TSM = \mathbb{R}F \oplus E^s \oplus E^u$. Recall also the weak (un)stable bundles $E^{\pm}$ (see Exercise 3.24) defined by $E^- := E^s \oplus \mathbb{R}F$, $E^+ := E^u \oplus \mathbb{R}F$.

The main theorem we wish to prove is the following, taken from [**DP07**]:

THEOREM 8.2. *Let (M, g) be a closed surface and ϕ_t an Anosov λ-geodesic flow on SM with infinitesimal generator F. Then*

$$V(x, v) \notin E^{\pm}(x, v) \quad \textit{for all} \quad (x, v) \in SM.$$

Despite the innocuous looking statement, proving this theorem will take some time. We will need the following auxiliary construction.

DEFINITION 8.3. Define a fibre bundle $\Lambda(SM)$ over SM as follows: given $(x, v) \in SM$, let

$$\Lambda_{(x,v)}(SM) := \left\{W \subseteq T_{(x,v)}SM \; : \; \dim W = 2, \; F(x, v) \in W\right\},$$

and then let $\Lambda(SM) = \coprod_{(x,v)\in SM} \Lambda_{(x,v)}(SM)$.

EXERCISE 8.4. Show that $\Lambda(SM)$ is a fibre bundle over SM, and compute its dimension.

We have two canonical sections $\mathcal{V}$ and $\mathcal{H}$ of the bundle $\Lambda(SM) \to SM$ defined by

$$\mathcal{V}(x, v) = \mathbb{R}V(x, v) \oplus \mathbb{R}F(x, v),$$
$$\mathcal{H}(x, v) = \mathbb{R}H(x, v) \oplus \mathbb{R}F(x, v).$$

We let Λ_V and Λ_H denote the images of $\mathcal{V}$ and $\mathcal{H}$ respectively. Λ_V is called the *Maslov cycle*. They are both codimension one submanifolds of $\Lambda(SM)$.

DEFINITION 8.5. The flow ϕ_t lifts naturally to a flow ϕ_t^* on $\Lambda(SM)$ by its differential, that is,

$$\phi_t^*(W) := d\phi_t(W).$$

We let F^* denote the infinitesimal generator of ϕ_t^*.

The key result we need in order to prove Theorem 8.2 is the following proposition. We do *not* require the Anosov property for this result.

PROPOSITION 8.6. *F^* is transversal to the Maslov cycle Λ_V.*

PROOF. We will define a smooth map $m : \Lambda(SM)\backslash\Lambda_H \to \mathbb{R}$ such that $\Lambda_V = m^{-1}(0)$. From the definition of m it will be clear that the statement F^* is transversal to Λ_V is equivalent to the statement $\dot{m}(0) \neq 0$. Then the proof will be completed by showing $\dot{m}(0) = 1$.

First, given $W \in \Lambda_{(x,v)}(SM)$ with $W \neq \mathbb{R}F(x, v) \oplus \mathbb{R}H(x, v)$, we define the unique real number $m(W)$ such that

$$m(W)H(x, v) + V(x, v) \in W.$$

In more detail, if we plot the $\{H, V\}$-plane of $T_{(x,v)}SM$ on the page, with V on the vertical axis and H on the horizontal axis, and the X-axis sticking out the page, then the cross section of W with the $\{H, V\}$-plane is a sloping line that is not horizontal. We let $\frac{1}{m(W)}$ denote the gradient of the line. Note that $W = \mathbb{R}F(x, v) \oplus \mathbb{R}V(x, v)$ if and only if $m(W) = 0$.

Now fix a point $(x_0, v_0) \in SM$, and define the smooth curve

$$m(t) := m\big(\phi_t^*(\mathcal{V}(x_0, v_0))\big).$$

Then it is clear that m satisfies the conditions stated above. We complete the proof therefore by showing $\dot{m}(0) = 1$.

Writing $H(t) := H(\phi_t(x_0, v_0))$ and $H = H(0) = H(x_0, v_0)$, and similarly for $V(t)$, V etc, $m(t)$ is the unique real number such that

$$m(t)H(t) + V(t) \in d\phi_t(\mathcal{V}(x_0, v_0)).$$

Since $\mathcal{V}(x, v) = \mathbb{R}V(x, v) \oplus \mathbb{R}F(x, v)$ and $d\phi_t(F(x, v)) = F(\phi_t(x, v))$, it follows that

$$m(t)H(t) + V(t) = a(t)F(t) + y(t)d\phi_t(V)$$

for some smooth functions $a(t)$, $y(t)$. Thus applying $d\phi_{-t}$ to both sides we obtain

$$m(t)d\phi_{-t}(H(t)) + d\phi_{-t}(V(t)) = a(t)F + y(t)V.$$

Now we differentiate with respect to t; using (4.2.1) we obtain

$$\dot{m}(t)d\phi_{-t}(H(t)) + m(t)d\phi_{-t}([F, H](t)) + d\phi_{-t}([F, V](t)) = \dot{a}(t)F + \dot{y}(t)V.$$

Now setting $t = 0$ and using the fact that

$$\begin{aligned} [F, V] &= [X + \lambda V, V] \\ &= [X, V] + [\lambda V, V] \\ &= -H + \lambda\,[V, V] - V(\lambda)V \\ &= -H - V(\lambda)(V), \end{aligned} \tag{8.1.1}$$

as well as the fact that $m(0) = 0$ we obtain

$$\dot{m}(0)H - H - V(\lambda)V = \dot{a}(0)F + \dot{y}(0)V.$$

Now since $\{X, H, V\}$ is a basis and F has no H-component, we can evaluate the coefficient of H on both sides to obtain

$$\dot{m}(0) - 1 = 0.$$

This completes the proof. □

DEFINITION 8.7. Proposition 8.6 implies that Λ_V determines an oriented codimension one cycle in $\Lambda(SM)$, and thus by duality a cohomology class

$$\mathfrak{m} \in H^1(\Lambda(SM), \mathbb{Z}).$$

Let E denote either E^- or E^+, and define $\nu \in H^1(SM, \mathbb{Z}) \cong \mathrm{Hom}(H_1(SM, \mathbb{Z}), \mathbb{Z})$ to be the cohomology class $\nu := E^*\mathfrak{m}$. Given a continuous closed curve $\gamma : S^1 \to SM$ define the *index* of γ to be $\nu([\gamma])$. Since E is ϕ_t-invariant, the previous result shows that if Γ is any closed orbit of ϕ_t then $\nu(\Gamma) \geq 0$.

8.2. Some algebraic topology

To progress further we need the following piece of algebraic topology.

PROPOSITION 8.8. *Let M be a closed orientable surface of genus* g. *If M is not diffeomorphic to the 2-torus then*

$$H_k(SM, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{2\mathrm{g}} & k = 2 \\ \mathbb{Z}^{2\mathrm{g}} \oplus \mathbb{Z}_{2\mathrm{g}-2} & k = 1 \\ \mathbb{Z} & k = 0, 3 \\ 0 & k \neq 0, 1, 2, 3. \end{cases}$$

PROOF. For this we need to recall the *Gysin sequence for homology* (see e.g. [**BT82**, Proposition 14.33]): if $\pi : N \to M$ is an oriented sphere bundle with fibre S^{n-1} then there is a long exact sequence in homology

$$\cdots \to H_{k-n+1}(M, \mathbb{Z}) \to H_k(N, \mathbb{Z}) \overset{\pi_*}{\to} H_k(M, \mathbb{Z}) \overset{\gamma}{\to} H_{k-n}(M, \mathbb{Z}) \to \dots,$$

Applying this to the unit sphere bundle of an orientable manifold M^n, we first note that the map $H_k(M,\mathbb{Z}) \stackrel{\gamma}{\to} H_{k-n}(M,\mathbb{Z})$ can only be non-trivial if both $H_k(M,\mathbb{Z})$ and $H_{k-n}(M,\mathbb{Z})$ are non-trivial, which can happen only when $k = n$ and M is compact. Hence when $n = 2$ and M is closed we obtain

$$H_1(M,\mathbb{Z}) \to H_2(SM,\mathbb{Z}) \to H_2(M,\mathbb{Z}) \stackrel{\gamma}{\to} H_0(M,\mathbb{Z}) \to H_1(SM,\mathbb{Z}) \stackrel{\pi_*}{\to} H_1(M,\mathbb{Z}) \to 0.$$

We have $H_2(M,\mathbb{Z}) = H_0(M,\mathbb{Z})$ and in this case the map γ is multiplication by the Euler characteristic $\chi(M)$, that is

$$0 \to H_1(M,\mathbb{Z}) \to H_2(SM,\mathbb{Z}) \to \mathbb{Z} \stackrel{\chi(M)}{\to} \mathbb{Z} \to H_1(SM,\mathbb{Z}) \stackrel{\pi_*}{\to} H_1(M,\mathbb{Z}) \to 0.$$

Now recall that if M has genus g we have

$$H_k(M,\mathbb{Z}) = \begin{cases} \mathbb{Z}^{2\mathrm{g}} & k = 1 \\ \mathbb{Z} & k = 0,2 \\ 0 & k \neq 0,1,2. \end{cases}$$

Thus the sequence becomes

$$0 \to \mathbb{Z}^{2\mathrm{g}} \to H_2(SM,\mathbb{Z}) \to \mathbb{Z} \stackrel{2-2\mathrm{g}}{\to} \mathbb{Z} \to H_1(SM,\mathbb{Z}) \stackrel{\pi_*}{\to} \mathbb{Z}^{2\mathrm{g}} \to 0.$$

Thus we conclude that if M is not diffeomorphic to the 2-torus then

$$H_k(SM,\mathbb{Z}) = \begin{cases} \mathbb{Z}^{2\mathrm{g}} & k = 2 \\ \mathbb{Z}^{2\mathrm{g}} \oplus \mathbb{Z}_{2\mathrm{g}-2} & k = 1 \\ \mathbb{Z} & k = 0,3 \\ 0 & k \neq 0,1,2,3, \end{cases}$$

where we have used the fact that since SM is necessarily orientable, $H_3(SM,\mathbb{Z}) \cong \mathbb{Z}$. □

COROLLARY 8.9. *If M is a closed orientable surface that is not diffeomorphic to the 2-torus and $\pi : SM \to M$ is the footpoint map then $\pi_* : H_1(SM,\mathbb{R}) \to H_1(M,\mathbb{R})$ is an isomorphism. Moreover the kernel of $\pi_* : H_1(SM,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ is precisely the torsion subgroup of $H_1(SM,\mathbb{Z})$.*

Using duality (or a similar argument with the Gysin sequence in cohomology) proves the following result, which we will use this in Chapter 11:

COROLLARY 8.10. *If M is a closed orientable surface that is not diffeomorphic to the 2-torus and $\pi : SM \to M$ is the footpoint map then $\pi^* : H^1(M,\mathbb{R}) \to H^1(SM,\mathbb{R})$ is an isomorphism.*

We now require the following classical theorem in Riemannian geometry; the reader is referred to [**Cha06**, Theorem IV.5.1] for a proof.

THEOREM 8.11. *Let (M,g) be a closed Riemannian manifold. Then every non-trivial free homotopy class of loops contains a closed geodesic.*

As a consequence we obtain the following corollary.

COROLLARY 8.12. *If M is a closed orientable surface of genus* $\mathrm{g} \geq 2$ *then every homology class in $H_1(M,\mathbb{Z})$ contains a closed geodesic.*

PROOF. This is immediate from the previous theorem, since as $\mathrm{g} \geq 2$, every homology class contains a homotopically non-trivial representative. □

REMARK 8.13. The corollary is also true for $M = S^2$, although there a different argument is needed; see for instance [**Kli78**].

8.3. Proving V is transversal to $E^{\pm}$

Returning to working towards the proof of Theorem 8.2, we finally use the Anosov condition.

LEMMA 8.14. *Returning to the situation in Definition 8.7, suppose that ϕ_t is in addition assumed to be Anosov. Then $\nu = 0 \in H^1(SM,\mathbb{Z})$.*

PROOF. The proof uses Ghys' Theorem 3.17. Recall that this tells us that if g_0 denotes a metric on M of constant curvature -1, and $\pi^0 : S^0M \to M$ the corresponding sphere bundle and $\psi_t : S^0M \to S^0M$ the corresponding geodesic flow, then there exists a homeomorphism $G : SM \to S^0M$ carrying orbits of ϕ_t into orbits of ψ_t. Note moreover that G determines an isomorphism $G_* : H_1(SM, \mathbb{Z}) \to H_1(S^0M, \mathbb{Z})$, which necessarily maps the torsion subgroup of $H_1(SM, \mathbb{Z})$ into the torsion subgroup of $H_1(S^0M, \mathbb{Z})$, and hence factors to define an isomorphism

$$\hat{G}_* : H_1(SM, \mathbb{Z})/\ker \pi_* \to H_1(S^0M, \mathbb{Z})/\ker \pi^0_*.$$

Since $\nu : H_1(SM, \mathbb{Z}) \to \mathbb{Z}$ is a homomorphism, it too factors through $\ker \pi_*$ to define a homomorphism $\hat{\nu} : H_1(SM, \mathbb{Z})/\ker \pi_* \to \mathbb{Z}$, and thus to show $\nu = 0$ it suffices to show that the homomorphism

$$\eta := \hat{\nu} \circ \hat{G}_*^{-1} \circ (\pi^0_*)^{-1} : H_1(M, \mathbb{Z}) \to \mathbb{Z}$$

is identically zero. But given $c \in H_1(M, \mathbb{Z})$, let $\gamma(t)$ denote a closed geodesic belonging to the homology class c. Then $\hat{G}_*^{-1} \circ (\pi^0_*)^{-1}(c)$ contains the curve $\Gamma(t) := G^{-1}(\gamma(t), \dot{\gamma}(t))$ as a representative, and thus by Proposition 8.6,

$$\eta(c) = \nu([\Gamma]) \geq 0.$$

Thus $\eta : H_1(M, \mathbb{Z}) \to \mathbb{Z}$ is a homomorphism such that $\eta(c) \geq 0$ for all $c \in H_1(M, \mathbb{Z})$. This of course implies $\eta = 0$, as was required.

□

REMARK 8.15. Let us also note that applying Corollary 3.33, we see that ϕ_t is transitive and hence the non-wandering set Ω of ϕ_t is all of SM. We will use this in the proof of Theorem 8.2 below.

We can now finally prove Theorem 8.2.

PROOF. *(of Theorem 8.2)*
As before let E denote either E^- or E^+. It is sufficient to show that $E(SM) \cap \Lambda_V = \emptyset$. Suppose this fails, so there exists $(x, v) \in SM$ such that $E(x, v) \cap \mathbb{R}V(x, v) \neq \{0\}$, that is, $V(x, v) \in E(x, v)$. Since F^* is transversal to Λ_V, there exists $\varepsilon > 0$ such that for $t \in (-\varepsilon, \varepsilon)\backslash\{0\}$ we have

$$d\phi_t(E(x, v) \cap \mathbb{R}V(\phi_t(x, v)) = E(\phi_t(x, v)) \cap \mathbb{R}V(\phi_t(x, v)) = \{0\}.$$

Now set $\theta_\pm := \phi_{\pm\varepsilon/2}(x, v)$. Then $E(\theta_\pm) \cap \Lambda_V = \{0\}$. We can then find neighborhoods $U_\pm$ of $\theta_\pm$ such that $E(U_\pm) \cap \Lambda_V = \{0\}$. Specifically, we claim that we can find $\eta \in U_+$ and $T > 0$ such that $\phi_T\eta \in U_-$. Admitting this for now, we can then find paths γ_+ from θ_+ to η and c_- from $\phi_T\eta$ to θ_- such that $\gamma_\pm$ is wholly contained in $U_\pm$. Now let γ denote the concatenated loop running from (x, v) to θ_+ via ϕ_t then along γ_+ to η, then down to $\phi_T\eta$ via ϕ_t and then along γ_- to θ_- and then back up (x, v) via ϕ_t. We claim that $\nu(\gamma) > 0$, thus contradicting Lemma 8.14.

Indeed, we have a $+1$ contribution at (x, v) to $\nu(\gamma)$, and along the ϕ_t portions the contribution is non-negative, and along the $\gamma_\pm$ sections the contribution is zero, since their images are contained in $U_\pm$.

It thus remains to deduce the existence of such η, T. More generally, let us first show that if $\theta \in \Omega$ and θ_1, θ_2 are points on orbit of θ with $\theta_2 = \phi_s\theta_1$ for some $s > 0$ then given neighborhoods U_i of θ_i there exists $\eta \in U_2$ and $T > 0$ such that $\phi_T\eta \in U_1$. For this note that since Ω is invariant, $\theta_2 \in \Omega$. Since $\phi_s(U_1) \cap U_2$ is an open set containing θ_2, there exists $\eta \in \phi_s(U_1) \cap U_2$ and $T' > s$ such that $\phi_{T'}\eta \in \phi_s(U_1) \cap U_2$. Then take $T = T' - s$, so $\phi_T\eta \in \phi_{-s}(\phi_s(U_1) \cap U_2) \subseteq U_1$. Finally, we use the fact that by Remark 8.15 above, $\Omega = SM$, and so this certainly works for $\theta = (x, v)$ and $\theta_1 = \theta_+, \theta_2 = \theta_-$. This completes the proof. □

REMARK 8.16. Note that the proof just given uses Theorem 3.17 only to conclude that $\Omega = SM$ and that there exists an orbit equivalence between ϕ_t and a geodesic flow ψ_t. In particular, Theorem 3.17 is not needed for the proof of Theorem 8.2 for the case where the flow ϕ_t is a geodesic flow (as opposed to a λ-geodesic flow).

8.4. The Riccati equation

Having finally now completed the proof of Theorem 8.2, we can reap the benefits.

DEFINITION 8.17. Theorem 8.2 implies the existence of functions $r^\pm$ on SM such that

$$H(x, v) + r^-(x, v)V(x, v) \in E^-(x, v),$$
$$H(x, v) + r^+(x, v)V(x, v) \in E^+(x, v).$$

Indeed, the picture is the same as the definition of m in the proof of Proposition 8.6, only now $r^{\pm}(x,v)$ represents the gradient of the line. Since $E^{-}(x,v) \cap E^{+}(x,v) = \{0\}$, we have

$$r^{-}(x,v) \neq r^{+}(x,v) \quad \text{for all } (x,v) \in SM.$$

REMARK 8.18. Since M is a surface, Theorem 11.4 below tells us that the bundles $E^{\pm}$ are C^1, and hence the functions $r^{\pm}$ are also C^1.

We now prove:

PROPOSITION 8.19. *Let $r = r^{\pm}$, and $E = E^{\pm}$. Then r satisfies the following equation:*

$$F(r) + r^2 + K - H(\lambda) + \lambda^2 - V(\lambda)r = 0. \tag{8.4.1}$$

If we restrict the equation above to a flow line of F, we obtain an ODE of Riccati-type. For the purposes of these notes we shall refer to (8.4.1) as 'the' *Riccati equation*. It makes sense as r is continuous since E is, and is smooth along ϕ_t (and so $F(r)$ makes sense) as E is ϕ_t-invariant.

Alternatively, if we let

$$\mathbb{K} := K - H(\lambda) + \lambda^2 + F(V(\lambda)), \tag{8.4.2}$$

then we can write this as

$$F(r - V(\lambda)) + r(r - V(\lambda)) + \mathbb{K} = 0. \tag{8.4.3}$$

The reason for introducing $\mathbb{K}$ will become clear in the next chapter (see Proposition 9.7).

PROOF. Pick $(x,v) \in SM$, and let $H(t) = H(\phi_t(x,v))$ and $H = H(0) = H(x,v)$ etc. Set

$$\eta(t) := d\phi_{-t}(H(t) + r(t)V(t)), \tag{8.4.4}$$

so $\eta(t) \in E$ for all t. Now differentiate (8.4.4) with respect to t and set $t = 0$; using (4.2.1) we obtain

$$\dot{\eta}(0) = [F, H] + F(r)V + r[F, V].$$

From (8.1.1) we know $[F, V] = -H - V(\lambda)V$. Next,

$$\begin{aligned}
[F, H] &= [X + \lambda V, H] \\
&= [X, H] + [\lambda V, H] \\
&= KV + \lambda[V, H] - H(\lambda)V \\
&= KV - \lambda X - H(\lambda)V \\
&= -\lambda F + (K - H(\lambda) + \lambda^2)V. \qquad (8.4.5)
\end{aligned}$$

Thus

$$\dot{\eta}(0) = -\lambda F + (K - H(\lambda) + \lambda^2)V + F(r)V - r(H + V(\lambda)V),$$

and noting that $H = \eta(0) - rV$, rearranging we have

$$\dot{\eta}(0) + r\eta(0) + \lambda F = \left\{r^2 + F(r) + K - H(\lambda) + \lambda^2 - V(\lambda)r\right\} V. \tag{8.4.6}$$

Now we observe that $\dot{\eta}(0) \in E$, since $\eta(t) \in E$ for all t. Thus the left-hand side of (8.4.6) lies in E, and hence so must the right-hand side. But since $V \notin E$ by Theorem 8.2 this forces the coefficient of the right-hand side of (8.4.6) to be zero, which is precisely what we wanted to show. □

CHAPTER 9

The Pestov identity and the proof of Theorem 7.15

In this chapter we will prove several integral identities, the most important of which is the Pestov identity (Proposition 9.8). This will then allow us to deduce Theorem 7.15.

9.1. Preliminaries

DEFINITION 9.1. Let (M, g) be a closed surface. The form $\Theta = -\alpha \wedge d\alpha = \alpha \wedge \beta \wedge \psi$ is a volume form on M, and Θ gives rise to the Liouville measure μ of Corollary *3.14*.

Recall that the *divergence* of a vector field Y with respect to Θ is the smooth function $\mathrm{div}_\Theta Y$ such that

$$L_Y\Theta = \mathrm{div}_\Theta Y \cdot \Theta.$$

LEMMA 9.2. *It holds that:*

$$L_X\Theta = L_H\Theta = L_V\Theta = 0,$$
$$\mathrm{div}_\Theta F = V(\lambda).$$

PROOF. This is very simple once we recall *Cartan's equation*: for any vector field Y and form ω,

$$L_Y\omega = i_Y d\omega + d i_Y \omega.$$

In particular when Θ is a top dimensional form,

$$L_Y\Theta = d i_Y\Theta.$$

Now since

$$\alpha \wedge \beta \wedge \psi(X, \cdot, \cdot) = \beta \wedge \psi(\cdot, \cdot),$$

we have

$$d i_X\Theta = d(\beta \wedge \psi) = d\beta \wedge \psi - \beta \wedge d\psi = 0$$

by the structural equations. Thus $L_X\Theta = 0$, and similarly $L_H\Theta = L_V\Theta = 0$. Now

$$\begin{aligned} L_F\Theta &= d i_F \Theta \\ &= d i_X\Theta + d i_{\lambda V}\Theta \\ &= 0 + d\lambda \wedge i_V\Theta + \lambda d i_V\Theta \\ &= d\lambda \wedge \alpha \wedge \beta. \end{aligned}$$

Now write

$$d\lambda = X(\lambda)\alpha + H(\lambda)\beta + V(\lambda)\psi$$

to observe that

$$d\lambda \wedge \alpha \wedge \beta = V(\lambda)\psi \wedge \alpha \wedge \beta = (-1)^2 V(\lambda)\Theta.$$

□

The next lemma is a version of integration by parts.

LEMMA 9.3. *Let N be a closed manifold, Θ a volume form on N, Y a vector field and $f \in C^\infty(N, \mathbb{R})$. Then*

$$\int_N Y(f)\Theta = -\int_N f L_Y\Theta. \tag{9.1.1}$$

If instead N has boundary ∂N, it holds that

$$\int_N Y(f)\Theta = -\int_N f L_Y\Theta + \int_{\partial N} f i_Y\Theta. \tag{9.1.2}$$

PROOF. First note

$$\begin{aligned} d(f i_Y \Theta) &= df \wedge i_Y \Theta + f d i_Y \Theta \\ &= df \wedge i_Y \Theta + f L_Y \Theta. \end{aligned}$$

Next note that

$$\begin{aligned} i_Y(df \wedge \Theta) &= i_Y(df)\Theta - df \wedge i_Y \Theta \\ &= Y(f)\Theta - df \wedge i_Y \Theta. \end{aligned}$$

But $i_Y(df \wedge \Theta) = 0$ as $df \wedge \Theta = 0$, and thus

$$d(f i_Y \Theta) = Y(f)\Theta + f L_Y \Theta.$$

Now apply Stokes' theorem to conclude:

$$0 = \int_N d(f i_Y \Theta) = \int_N Y(f)\Theta + \int_N f L_Y \Theta.$$

□

REMARK 9.4. Compare with the definition of flow cohomology (see Definition 6.21). We see now that if $f : SM \to \mathbb{R}$ is cohomologous to zero, that is, there exists $u : SM \to \mathbb{R}$ such that $X(u) = f$ then

$$\int_{SM} f\Theta = 0.$$

COROLLARY 9.5. *There do not exist Anosov geodesic flows on the sphere or the torus.*

PROOF. Let M be a closed surface. Take $\lambda = 0$, and $r = r^{\pm}$, so Proposition 8.19 gives $X(r) + r^2 + K = 0$. Now integrate both sides over SM with respect to $d\mu$:

$$\int_{SM} X(r)d\mu + \int_{SM} r^2 d\mu + \int_{SM} K d\mu = 0. \tag{9.1.3}$$

Consider the third term in (9.1.3). Note that $\alpha \wedge \beta = \pi^*\Omega_a$, where Ω_a is the area form (see Definition 4.11). Hence we can write

$$\int_{SM} K d\mu = \int_{S_x M} d\theta \int_M K d\Omega_a = 2\pi \cdot 2\pi\chi(M) = 4\pi^2\chi(M), \tag{9.1.4}$$

where we have used the Gauss-Bonnet theorem. The first term of (9.1.3) is zero by Lemma 9.2 and Lemma 9.3 (although it is in fact the case that r is differentiable (see Remark 8.18), even if we only knew r was continuous this would be okay, since the previous lemma will still hold if the function f is only assumed continuous and differentiable along Y, by an approximation argument. Alternatively one could use Lemma 10.28 below.).

We conclude that

$$0 \le \int_{SM} r^2 d\mu = -4\pi^2\chi(M).$$

In fact we have

$$\int_{SM} r^2 d\mu > 0,$$

since if $\int_{SM} r^2 = 0$ then $r \equiv 0$, but note r is either r^- or r^+, and we know $r^-(x, v) \neq r^+(x, v)$ for all $(x, v) \in SM$. This completes the proof. □

REMARK 9.6. The corollary also follows easily from Theorem 3.17. However the proof given is independent of Theorem 3.17, see Remark 8.16.

9.2. The first integral identity

For the remainder of this chapter, let (M, g) denote a closed surface. In order to prove Theorem 7.15 we need three integral identities: Proposition 9.7, Proposition 9.9 and Proposition 9.12. Here is the first.

PROPOSITION 9.7. *Let $s : SM \to M$ be a smooth function and suppose ϕ_t is Anosov. Then for $r = r^{\pm}$ and $\mathbb{K}$ defined as in* (8.4.2),

$$\int_{SM} [F(s)]^2 \, d\mu - \int_{SM} \mathbb{K}s^2 d\mu = \int_{SM} [F(s) - rs + sV(\lambda)]^2 \, d\mu,$$

with equality if and only if $s \equiv 0$. In particular,

$$\int_{SM} [F(s)]^2 \, d\mu - \int_{SM} \mathbb{K}s^2 d\mu \geq 0,$$

with equality if and only if $s \equiv 0$.

PROOF. We begin by expanding $[F(s) - rs + sV(\lambda)]^2$:

$$\begin{aligned} [F(s) - rs + sV(\lambda)]^2 &= [F(s)]^2 + s^2r^2 + s^2 [V(\lambda)]^2 \\ &- 2F(s)sr + 2F(s)\, sV(\lambda) - 2s^2rV(\lambda). \end{aligned}$$

Using (8.4.3) we can rewrite this as

$$(9.2.1) \qquad [F(s) - rs + sV(\lambda)]^2 = [F(s)]^2 - \mathbb{K}s^2 - \left\{F((r - V(\lambda))s^2) - s^2[V(\lambda)]^2 + s^2rV(\lambda)\right\}.$$

Now we use Lemma 9.2 and Lemma 9.3 to conclude that

$$\int_{SM} F((r - V(\lambda))s^2) d\mu = - \int_{SM} ((r - V(\lambda))s^2)V(\lambda) d\mu,$$

and the three $\{\cdot\}$ bracketed terms in (9.2.1) disappear when we integrate, which gives

$$\int_{SM} (Fs)^2 d\mu - \int_{SM} \mathbb{K}s^2 d\mu = \int_{SM} [F(s) - rs + sV(\lambda)]^2 \, d\mu,$$

as we want.

Finally if

$$[F(s) - rs + sV(\lambda)]^2 = 0,$$

then since we can have $r = r^-$ and $r = r^+$, we obtain

$$(r^- - r^+)s = 0,$$

and then since $r^-(x, v) \neq r^+(x, v)$ for all $(x, v) \in SM$ it follows that $s \equiv 0$. □

9.3. The Pestov identity and the second integral identity

It is hard to overstate the importance of the next result, called the *Pestov identity*. Although it looks somewhat ugly, when we integrate it over SM below, lots of the terms disappear, and the statement becomes much more concise.

PROPOSITION 9.8. *For every smooth function $u : SM \to \mathbb{R}$ we have*

$$\begin{aligned} 2H(u) \cdot VF(u) &= [F(u)]^2 + [H(u)]^2 - \left(K - H(\lambda) + \lambda^2\right) [V(u)]^2 \\ &\quad + F(H(u) \cdot V(u)) + V(\lambda) \cdot H(u) \cdot V(u) - H(F(u) \cdot V(u)) \\ &\quad + V(F(u) \cdot H(u)). \end{aligned}$$

PROOF. We first recall the brackets for the basis $\{F, H, V\}$. We have

$$\begin{aligned} [V, F] &= H + V(\lambda)V, \\ [H, V] &= F + \lambda V, \\ [F, H] &= -\lambda F + (K - H(\lambda) + \lambda^2)V, \end{aligned}$$

where we have used (8.1.1) and (8.4.5) and Lemma 7.4.

Let us begin by looking at

$$\begin{aligned} 2H(u)\cdot VF(u) - V(F(u)\cdot H(u)) &= H(u)\cdot VF(u) - VH(u)\cdot F(u) \\ &= H(u)\cdot\{FV(u) + [V,F](u)\} \\ &\quad -Fu\{HV(u) + [V,H](u)\} \\ &= Hu\{FV(u) + H(u) + V(\lambda)\cdot V(u)\} \\ &\quad -F(u)\{HV(u) - F(u) + \lambda V(u)\} \\ &= [F(u)]^2 + [H(u)]^2 + FV(u)\cdot H(u) \\ &\quad -HV(u)\cdot F(u) \\ &\quad -\lambda F(u)\cdot V(u) + H(u)\cdot V(\lambda)\cdot V(u). \end{aligned}$$

Now using the fact that

$$F(V(u)\cdot H(u)) = FV(u)\cdot H(u) + V(u)\cdot FH(u)$$

and that

$$H(F(u)\cdot V(u)) = HF(u)\cdot V(u) + F(u)\cdot HV(u),$$

together with

$$[F,H](u)\cdot V(u) = FH(u)\cdot V(u) - HF(u)\cdot V(u),$$

we can rewrite this last equation as

$$\begin{aligned} 2H(u)\cdot VF(u) - V(F(u)\cdot H(u)) &= [F(u)]^2 + [H(u)]^2 + F(V(u)\cdot H(u)) \\ &\quad -H(V(u)\cdot F(u)) - [F,H](u)\cdot V(u) \\ &\quad -\lambda F(u)\cdot V(u) + H(u)\cdot V(\lambda)\cdot V(u), \end{aligned}$$

and then finally substituting for $[F,H]$ the extraneous terms cancel and we obtain

$$\begin{aligned} 2H(u)\cdot VF(u) - V(F(u)\cdot H(u)) &= [F(u)]^2 + [H(u)]^2 \\ &\quad -(K - H(\lambda) + \lambda^2)\,[V(u)]^2 \\ &\quad +F(H(u)\cdot V(u)) \\ &\quad +V(\lambda)\cdot H(u)\cdot V(u) - H(V(u)\cdot F(u)), \end{aligned}$$

which completes the proof. □

We can now prove the second of the two integral identities we need.

COROLLARY 9.9. *The integral Pestov identity:*

$$(9.3.1)\qquad \begin{aligned} 2\int_{SM} H(u)\cdot VF(u)d\mu &= \int_{SM}[F(u)]^2\,d\mu + \int_{SM}[H(u)]^2\,d\mu \\ &\quad -\int_{SM}(K - H(\lambda) + \lambda^2)\,[V(u)]^2\,d\mu. \end{aligned}$$

PROOF. We integrate the Pestov identity over SM and note that by Lemma 9.2 and Lemma 9.3, we have

$$\int_{SM} H(F(u)\cdot V(u))d\mu = \int_{SM} V(F(u)\cdot H(u))d\mu = 0,$$

since $L_V\Theta = L_H\Theta = 0$ and moreover

$$\int_{SM} F(H(u)\cdot V(u))d\mu = -\int_{SM}(H(u)\cdot V(u))\cdot V(\lambda)d\mu,$$

since $\mathrm{div}_\Theta F = V(\lambda)$ which disposes of the remaining two terms. □

Let us also give a version of this corollary in the case where M is compact with boundary ∂M; we will use this result in Chapter 12.

EXERCISE 9.10. Suppose M has boundary ∂M. Prove the integral Pestov identity:

$$
\begin{aligned}
2\int_{SM} H(u)\cdot VF(u)d\mu &= \int_{SM}[F(u)]^2\,d\mu + \int_{SM}[H(u)]^2\,d\mu \\
&\quad - \int_{SM}(K - H(\lambda) + \lambda^2)\,[V(u)]^2\,d\mu \\
&\quad + \int_{\partial(SM)} H(u)\cdot V(u)\cdot i_F\Theta - \int_{\partial(SM)} Fu\cdot Vu\cdot i_H\Theta \\
&\quad - \int_{\partial(SM)} Fu\cdot Hu\cdot i_V\Theta.
\end{aligned}
$$

To show how far we have come, let us now quickly prove a special case of Theorem 1.9 (which is itself a special case of Theorem 7.15).

THEOREM 9.11. *Let (M,g) be a closed surface of negative curvature and let $h \in C^\infty(M,\mathbb{R})$ a function such that $I[h] = 0$. Then $h \equiv 0$.*

PROOF. This is easy, given what we have already proved. The Livsic Cocycle Regularity Theorem 6.11 gives the existence of a smooth $u \in C^\infty(SM,\mathbb{R})$ such that $X(u) = h$, and then since $VX(u) = V(u) = 0$ as V is the vertical vector field, viewing h as a function $SM \to \mathbb{R}$ sending $(x,v) \mapsto h(x)$ we clearly have $V(h) = 0$ since h is independent of v. Thus Corollary 9.9 reduces to

$$\int_{SM}[X(u)]^2\,d\mu + \int_{SM}[H(u)]^2\,d\mu - \int_{SM} K\,[V(u)]^2\,d\mu = 0.$$

Since $K < 0$ each term is non-negative, and thus we conclude all three terms are zero; in particular $X(u) = h = 0$. ☐

9.4. The third integral identity

Now we prove the final integral identity.

PROPOSITION 9.12. *Let $\mathbb{K}$ be defined by* (8.4.3). *Then for any $u \in C^\infty(SM,\mathbb{R})$,*

$$\int_{SM}[FV(u)]^2\,d\mu - \int_{SM}\mathbb{K}\,[V(u)]^2\,d\mu = \int_{SM}[VF(u)]^2\,d\mu - \int_{SM}[F(u)]^2\,d\mu.$$

PROOF. By (8.1.1) we have

$$FV(u) = VF(u) - H(u) - V(\lambda)\cdot V(u),$$

and thus squaring both sides we have

$$
\begin{aligned}
[FV(u)]^2 &= [VF(u)]^2 + [H(u)]^2 + [V(\lambda)]^2\cdot[V(u)]^2 \\
&- 2VF(u)\cdot H(u) - 2VF(u)\cdot V(\lambda)\cdot V(u) \\
&\quad +2V(\lambda)\cdot V(u)\cdot H(u).
\end{aligned}
$$

Then using the fact that

$$2FV(u)\cdot V(\lambda)\cdot V(u) + 2V(\lambda)\cdot V(u)\cdot H(u) = 2VF(u)\cdot V(\lambda)\cdot V(u) - 2\,[V(\lambda)]^2\cdot[V(u)]^2\,,$$

by (8.1.1) again, and the fact that

$$F\left(V(\lambda)\cdot[V(u)]^2\right) = 2V(\lambda)\cdot V(u)\cdot FV(u) + [V(u)]^2\,F(V(\lambda)),$$

we see that

$$
\begin{aligned}
[FV(u)]^2 &= [VF(u)]^2 + [H(u)]^2 - [V(\lambda)]^2\cdot[V(u)]^2\,. \\
&- 2VF(u)\cdot H(u) - F\left(V(\lambda)\cdot[V(u)]^2\right) + [V(u)]^2\,F(V(\lambda)).
\end{aligned}
$$

Integrating this last expression we obtain

$$
\begin{aligned}
2\int_{SM} H(u)\cdot VF(u)d\mu &= \int_{SM}[VF(u)]^2\,d\mu + \int_{SM}[H(u)]^2\,d\mu \\
&\quad - \int_{SM}[FV(u)]^2\,d\mu + \int_{SM}[V(u)]^2\,F(V(\lambda))d\mu,
\end{aligned}
\tag{9.4.1}
$$

since

$$\int_{SM} F\left(V(\lambda)\cdot[V(u)]^2\right)d\mu = -\int_{SM}[V(\lambda)]^2\cdot[V(u)]^2\,d\mu$$

by Lemma 9.2 and Lemma 9.3. Now we combine (9.4.1) and Corollary 9.9 to obtain the desired identity. □

9.5. Proving the theorem

We will use the following exercise in the forthcoming proof.

EXERCISE 9.13. The measure μ is invariant under $v \mapsto e^{it}v$ for any $t \in \mathbb{R}$.

PROOF. *(of Theorem 7.15)*

All the elements are in place, and the proof is painless. The Livsic Cocycle Regularity Theorem 6.11 gives the existence of a smooth function $u : SM \to \mathbb{R}$ that satisfies the cohomological equation

$$F(u) = h + \theta.$$

Lemma 6.22 (which is clearly still valid in the more general setting of Theorem 7.15) shows that it is enough to show that u is independent of v, that is, actually $u : M \to \mathbb{R}$, or equivalently, we must show $V(u) = 0$. Applying Proposition 9.7 with $s = V(u)$ and Proposition 9.12, we have

$$0 \le \int_{SM}[FV(u)]^2 - \int_{SM}\mathbb{K}[V(u)]^2 = \int_{SM}[VF(u)]^2\,d\mu - \int_{SM}[F(u)]^2\,d\mu, \tag{9.5.1}$$

with equality if and only if $V(u) = 0$. Since $F(u) = h + \theta$,

$$\begin{aligned} VF(u) &= V(h+\theta) \\ &= V(\theta) \\ &= \frac{d}{ds}\Big|_{s=0}\theta_x(e^{is}v), \\ &= \theta_x(\mathtt{i}v), \end{aligned}$$

and

$$[F(u)]^2 = h^2 + 2h\theta + \theta^2,$$

and hence the right-hand side of (9.5.1) becomes

$$\int_{SM}[\theta_x(\mathtt{i}v)]^2\,d\mu - \int_{SM}h^2 d\mu - 2\int_{SM}h\theta d\mu - \int_{SM}[\theta_x(v)]^2\,d\mu.$$

By Exercise 9.13 the measure μ is invariant under $v \mapsto -v$ and $v \mapsto \mathtt{i}v$ and hence we have

$$\int_{SM}[\theta_x(\mathtt{i}v)]^2\,d\mu = \int_{SM}[\theta_x(v)]^2\,d\mu,$$

and for any 1-form η,

$$\int_{SM}\eta d\mu = 0,$$

since η_x changes sign as $v \mapsto -v$; in particular

$$\int_{SM}h\theta d\mu = 0.$$

Thus (9.5.1) becomes

$$0 \le \int_{SM}[FV(u)]^2 - \int_{SM}\mathbb{K}[V(u)]^2 = -\int_{SM}h^2 d\mu \le 0,$$

and so $V(u) = 0$ and the proof is complete. □

CHAPTER 10

Entropy

In this chapter we discuss applications of Theorem 7.15 to entropy (specifically, *entropy production*). However in an effort to make these notes more self-contained, the first four sections and Section 6 are devoted to background material on various aspects of entropy.

We begin with a discussion on when a flow preserves a volume form. We then define entropy, and in the third section we study a special invariant measure called the SRB measure. We then move onto Ruelle's notion of entropy production, which we relate to the Hausdorff dimension of the SRB measure, and also to λ-geodesic flows on closed surfaces; as a corollary we obtain an explicit description of the SRB potential in terms of the function r^+ from Definition 8.17.

10.1. Preserving volume forms

Let $\phi_t : N \to N$ be a flow on a closed N with infinitesimal generator F. Recall that ϕ_t is *volume preserving* with respect to a volume form ω if for any compact domain of integration $K \subseteq N$,

$$\mathrm{Vol}(\phi_t(K)) = \int_{\phi_t[K]} \omega = \int_K \phi_t^* \omega = \int_K \omega = \mathrm{Vol}(K).$$

Recall that the *divergence* of a vector field Y with respect to a volume form ω is the smooth function $\mathrm{div}_\omega Y$ defined by

$$\mathrm{div}_\omega Y \cdot \omega = L_Y \omega.$$

EXERCISE 10.1. Show that ϕ_t is volume preserving with respect to ω if and only if $\mathrm{div}_\omega F = 0$.

Question: When does ϕ_t preserve a volume form? In order to answer this, let us fix a 'background' volume form ω on N. Then any other volume form has the form $f\omega$ for some $f \in C^\infty(N)$, $f \neq 0$. Without loss of generality, below we will assume $f > 0$.

LEMMA 10.2. *It holds that*

$$\mathrm{div}_{f\omega} F = F(\log f) + \mathrm{div}_\omega F.$$

PROOF. We compute that

$$\begin{aligned} L_F(f\omega) &= d i_F(f\omega) + i_F d(f\omega) \\ &= df \wedge i_F\omega + f d i_F \omega \\ &= df \wedge i_F\omega + f L_F \omega \\ &= df \wedge i_F\omega + f(\mathrm{div}_\omega F)\omega. \end{aligned}$$

Now since $df \wedge \omega = 0$ as ω is top dimensional, we have

$$\begin{aligned} 0 &= i_F(df \wedge \omega) \\ &= i_F df \wedge \omega - df \wedge i_F \omega \\ &= F(f)\omega - df \wedge i_F\omega, \end{aligned}$$

and thus we have

$$L_F(f\omega) = (F(f) + f\mathrm{div}_\omega F)\omega.$$

Since f is strictly positive we can rewrite this as

$$\mathrm{div}_{f\omega} F = \frac{F(f)}{f} + \mathrm{div}_\omega F,$$

or alternatively

$$\mathrm{div}_{f\omega} F = F(\log f) + \mathrm{div}_\omega F. \tag{10.1.1}$$

□

This allows us to answer our question. Recall that a function $f \in C^\infty(N, \mathbb{R})$ is a coboundary with respect to F (in the sense of flow cohomology) if and only if $f = F(g)$ for some $g \in C^\infty(N, \mathbb{R})$ (see Definition 6.21).

COROLLARY 10.3. *Let $\phi_t : N \to N$ be a flow on a closed manifold N, with infinitesimal generator F. Then ϕ_t preserves a volume form if and only if* $\mathrm{div}_\omega F$ *is a coboundary for* some *volume form ω.*

PROOF. We have just shown that if ω and $f\omega$ are any two volume forms then $\mathrm{div}_{f\omega} F$ is flow cohomologous to $\mathrm{div}_\omega F$, since $\mathrm{div}_{f\omega} F - \mathrm{div}_\omega F = F(\log f)$. Hence if F preserves a volume form ω then for every volume form $f\omega$ we have $\mathrm{div}_{f\omega} F = F(\log f)$. Conversely if $\mathrm{div}_\omega F = F(f)$ then $\mathrm{div}_{e^{-f}\omega} F = 0$, since

$$\mathrm{div}_{e^{-f}\omega} F = F(\log e^{-f}) + \mathrm{div}_\omega F = F(-f) + F(f) = 0.$$

□

We now return to the situation we are most interested in. Let us introduce the following definition, which is the most general sort of flow we like to study.

DEFINITION 10.4. Let (M, g) denote a closed surface. Fix $h \in C^\infty(M, \mathbb{R})$ and $\theta \in \Omega^1(M)$. Define $\lambda : SM \to \mathbb{R}$ by

$$\lambda(x, v) = h(x) + V(x, v)(\theta_x(v)),$$

where V is as always the vertical vector field. This flow ϕ_t is the obtained from the superpositive of a magnetic field and a thermostat: recall from Lemma 7.7 and Definition 7.8 that this is modeling the effect of a magnetic field $h\Omega_a$ (where Ω_a is the area form, see Definition 4.11) and a thermostat with external field $E = \hat{g}\theta$. We will refer to ϕ_t as a *magnetic thermostat flow* determined by the pair (h, θ).

EXERCISE 10.5. Recall from Definition 9.1 that we have a preferred choice of volume form $\Theta = \alpha \wedge \beta \wedge \psi$ on SM. Suppose F is the infinitesimal generator of a magnetic thermostat flow ϕ_t determined by the pair (h, θ). Show that

$$\mathrm{div}_\Theta F = -\theta.$$

In fact, we can say more. Here is our first application of Theorem 7.15.

THEOREM 10.6. *An Anosov magnetic thermostat flow ϕ_t determined by the pair (h, θ) is volume preserving with respect to* some *volume form if and only if θ is exact.*

PROOF. If $\theta = dg$ then writing $F(g)$ for what should really be $F(g \circ \pi)$, we have

$$F(g) = dg = \theta = -\mathrm{div}_\Theta F,$$

and hence ϕ_t preserves a volume form by Corollary 10.3. For the converse, by Corollary 10.3 again, if ϕ_t is volume preserving with respect to a volume form ω then $\mathrm{div}_\omega F = 0$. Thus if $\Theta = f\omega$ then

$$\mathrm{div}_\Theta F = F(\log f) = -\theta.$$

But using Theorem 7.15, if the cohomological equation $F(u) = -\theta$ has a solution $u \in C^\infty(SM, \mathbb{R})$ then θ must be exact. □

10.2. Entropy

We now move on to discussing entropy. We will define two different types of entropy associated to a flow ϕ_t, and state the *Variational Principle* (Theorem 10.16), which relates the two. A beautiful comprehensive reference for all this material is [**Wal82**]. Another good reference is [**KH95**, Chapter 4], from which most of this material is taken from.

DEFINITION 10.7. Let $\phi_t \colon N \to N$ be a flow on a closed manifold N. A measure m on N is ϕ_t-*invariant* (or simply *invariant* if the flow ϕ_t is understood) if for any Borel set $B \subseteq N$, we have $m(\phi_t(B)) = m(B)$. In other words, if $\phi_{t*}m$ is the measure defined by $\phi_{t*}m(B) := m(\phi_t(B))$ then m is invariant if $\phi_{t*}m = m$ for all $t \in \mathbb{R}$. An invariant measure is called *ergodic* if any Borel invariant set has measure zero or one. Let $\mathcal{M}(\phi)$ denote the set of all ϕ_t-invariant Borel probability measures m on M. It can be shown that $\mathcal{M}(\phi)$ is a nonempty compact convex set (in the weak $*$-topology) of the compact set $\mathcal{M}$ of all Borel probability measures on N (the extreme points are the ergodic measures) - see for instance, [**KH95**, Section 4.1].

As an example of an invariant Borel probability measure, if Γ is a closed orbit of ϕ_t with period T then we can define a measure $m_\Gamma \in \mathcal{M}(\phi)$ by

$$\int_N f dm_\Gamma := \frac{1}{T}\int_0^T f(\phi_t x_0)dt,$$

where x_0 is some point in Γ. We call m_Γ the *δ-measure* of the closed orbit Γ.

DEFINITION 10.8. A *partition* of N is a disjoint finite collection $\alpha = \{A_1, \dots, A_k\}$ of Borel sets whose union is all of N. If $\alpha = \{A_1, \dots, A_k\}$ and $\beta = \{B_1, \dots, B_n\}$ are two partitions then the *join* $\alpha \vee \beta$ is the partition given by

$$\alpha \vee \beta = \{A_i \cap B_j \ : \ i = 1, \dots, k, \ j = 1, \dots, n\}.$$

Similarly we can form the join $\bigvee_i \alpha_i$ of multiple partitions $\alpha_1, \dots, \alpha_\ell$.

DEFINITION 10.9. Given a Borel probability measure m, the *entropy of a partition* $\alpha = \{A_1, \dots, A_k\}$ *with respect to* m is given by

$$H_m(\alpha) := -\sum_{i=1}^{k} m(A_i)\log m(A_i),$$

where $0\log 0$ is interpreted to be 0.

Let $\alpha = \{A_1, \dots, A_k\}$ a partition. Define $\phi_{-n}\alpha$ to be the partition

$$\phi_{-n}\alpha = \{\phi_{-n}(A_1), \dots, \phi_{-n}(A_k)\}.$$

DEFINITION 10.10. Let $m \in \mathcal{M}(\phi)$. The *metric entropy of ϕ_t with respect to m* is defined to be

$$h_m(\phi) := \sup_\alpha \lim_{n\to\infty} \frac{1}{2} H_m\left(\bigvee_{i=0}^{n-1} \phi_{-i}\alpha\right),$$

where the supremun is taken over all partitions α.

The following easy exercise gives an alternative definition of $h_m(\phi)$.

EXERCISE 10.11. Given a partition α, and $x \in N$, let $C_\alpha(x)$ denote the unique element of α with $x \in C_\alpha(x)$. Given $m \in \mathcal{M}(\phi)$, define the *information function* of m and α, written $I(m;\alpha)$ by

$$I(m;\alpha)(x) := -\log m(C_\alpha(x)).$$

Prove that

$$H_m(\alpha) = \int_N I(m,\alpha)dm,$$

and hence that

$$h_m(\phi) = \sup_\alpha \lim_{n\to\infty} \frac{1}{2}\int_N I(m;\phi_{-n}\alpha)dm.$$

We will now define a related concept; the *topological entropy of ϕ_t*.

DEFINITION 10.12. Select an arbitrary Riemannian metric g on N, and let $d = d_g$ denote the geodesic distance function on (N, g) from (2.1.1). Given $T > 0$ define a new metric d_T by

$$d_T(x, y) := \max_{t\in[0,T]} d(\phi_t x, \phi_t y).$$

Since N is compact, we can define $N(T, \varepsilon)$ to be the minimal (finite) number of balls of radius $\varepsilon > 0$ required to cover all of N under the d_T metric.

We define the *topological entropy of ϕ_t* to be the quantity

$$h_{\text{top}}(\phi) := \lim_{\epsilon\to 0}\limsup_{T\to\infty} \frac{1}{T}\log N(T, \varepsilon).$$

An easy argument shows that $h_{\text{top}}(\phi)$ is well defined:

LEMMA 10.13. *$h_{\text{top}}(\phi)$ is independent of the choice of the Riemannian metric g*

PROOF. Let g and g' be two Riemannian metrics on N, and let d and d' denote the corresponding geodesic metrics on N. Since the geodesic metric always induces the manifold topology, both d and d' define the same topology on N.

Now consider the set $D(\varepsilon) \subseteq N \times N$ consisting of all pairs (x, y) with $d(x, y) \geq \varepsilon$. Then $D(\varepsilon)$ is a compact subset of $N \times N$, and hence the continuous function d' attains its minimum $\delta(\varepsilon) > 0$ on $D(\varepsilon)$. Hence if $d'(x, y) < \delta(\varepsilon)$ then $d(x, y) < \varepsilon$, and so any $\delta(\varepsilon)$-ball in the metric d' is contained in an ε-ball in the metric d. This argument also works for the metrics d_T and d'_T, and so we conclude

$$N'(T, \delta(\varepsilon)) \geq N(T, \varepsilon),$$

(where $N'(T, \delta(\varepsilon))$ means with respect to d' etc.) and so

$$\limsup_{T\to\infty} \frac{1}{T} \log N'(T, \delta(\varepsilon)) \geq \limsup_{T\to\infty} \frac{1}{T} \log N(T, \varepsilon).$$

Letting $\varepsilon \to 0$ we see that $h'_{\text{top}}(\phi) \geq h_{\text{top}}(\phi)$. Interchanging g and g' completes the proof. □

EXERCISE 10.14. Here are two alternative characterizations of $h_{\text{top}}(\phi)$. Say that a set $Y \subseteq N$ is (T, ε)-*separated* if for any two points $x, y \in Y$ it holds that

$$d_T(x, y) > \varepsilon.$$

Then if $E(T, \varepsilon)$ denotes the maximum cardinality of a (T, ε)-separated set, prove that

$$h_{\text{top}}(\phi) = \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\log\, E(T, \varepsilon)}{T}. \tag{10.2.1}$$

Similarly, say that a set $Z \subseteq N$ is a (T, ε)-*spanning set* if for any point $x \in N$ there exists a point $y \in Z$ such that

$$d_T(x, y) < \varepsilon.$$

Now let $A(T, \varepsilon)$ denote the minimal cardinality of a (T, ε)-spanning set. Prove that

$$h_{\text{top}}(\phi) = \lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\log\, A(T, \varepsilon)}{T}.$$

EXERCISE 10.15. Let $\phi_t : N \to N$ be a flow on a closed manifold N, and consider the 'backward' flow $\psi_t := \phi_{-t}$. Show that for any invariant measure m

$$h_m(\phi) = h_m(\psi).$$

How are the metric and topological entropy related? The metric entropy $h_m(\phi)$ should be thought of as a quantitative estimate of the complexity of the flow ϕ_t from the point of view of the invariant measure m. It can be shown that the metric entropy of the union of two invariant sets is the sum (suitably weighted by m) of the entropy of each of the sets individually, and so in some sense the metric entropy measures the average complexity of ϕ_t. Meanwhile the topological entropy measures the global maximal complexity of ϕ_t. It thus makes sense to assume that the metric entropy is at most the topological entropy, with the metric entropy maximized for measures that assign the most weight to areas of high complexity.

This is indeed what happens; this is called the *Variational Principle*:

THEOREM 10.16. *Let $\phi_t : N \to N$ denote a flow on a closed manifold N. Then*

$$h_{\text{top}}(\phi) = \sup_{m\in\mathcal{M}(\phi)} h_m(\phi).$$

Moreover measures of maximum entropy, *that is, measures $m \in \mathcal{M}(\phi)$ such that $h_m(\phi) = h_{\text{top}}(\phi)$ always exist.*

The first statement of the theorem is actually true for any continuous flow on a compact metric space; for a proof of this see [**KH95**, Theorem 4.5.3]. The second statement is due to Newhouse; see [**New89**], and is one of the only places in these notes where the flow ϕ_t is actually required to be C^∞ (recall all flows are assumed smooth in these notes), e.g. class C^{2010} is not good enough. Section 4.5 of [**KH95**] also contains a much fuller discussion on the difference between metric entropy and topological entropy.

10.3. The SRB measure

In this subsection we define a special class of invariant measures, known as *SRB measures*. The 'SRB' stands for Sinai, Ruelle and Bowen, who did the pioneering work on this material; see Theorem 10.26 below. Good references for this section are [**KH95**, Chapter 20], [**Wal82**, Chapter 9] and the survey article [**You02**].

Let N be a closed manifold. Fix a Riemannian metric g on N, and let ω denote the volume form associated to g. Recall that a probability measure m on N is *absolutely continuous* if in any local chart it is given by integrating a density, that is, if

$$m(U) = \int_U f\omega,$$

where $f : N \to \mathbb{R}$ is a measurable function that is non-negative almost everywhere with respect to the Riemannian measure (f is called the *density*) and ω is a volume form (remember the manifold N is always assumed to be orientable). It is *positive* if the density is almost everywhere positive. A measure is called *smooth* if is absolutely continuous and the density f is positive and smooth.

THEOREM 10.17. *Let $\phi_t : N \to N$ be an Anosov flow on a closed Riemannian manifold (N, g). Then ϕ_t has at most one absolutely continuous invariant measure.*

In particular, ϕ_t has at most one smooth measure. A proof of this weaker statement can be found in [**KH95**, Theorem 20.4.1]. The full statement of Theorem 10.17 follows from Theorems 10.20 and 10.26 below.

Let $\phi_t : N \to N$ be an Anosov flow on a closed manifold N. There is a unique ϕ_t-invariant probability measure ρ on N called the *SRB measure*, which we will define below. In order to motivate this definition, we remark that *if* ϕ_t admits an absolutely continuous invariant measure (necessarily unique, by Theorem 10.17), then this invariant measure is precisely the SRB measure (see Theorem 10.26 below). However *not* all Anosov flows admit absolutely continuous invariant measures, and so we cannot take this as our definition of the SRB measure; it is too restrictive. In order to give the proper definition, we will first need to define *pressure*. This is a generalization of entropy, and, as with entropy, comes in two flavors. Let us begin with the case where $\phi_t : N \to N$ is any flow on N.

DEFINITION 10.18. Given a continuous function $f \in C^0(N, \mathbb{R})$ and an invariant measure $m \in \mathcal{M}(\phi)$ define the *metric pressure of ϕ_t with respect to f and m* by

$$P_m(\phi; f) := h_m(\phi) + \int_N f dm.$$

Next we have the topological pressure.

DEFINITION 10.19. Given $f \in C^0(N, \mathbb{R})$ and $\varepsilon > 0$, let

$$S(\phi, f, T, \varepsilon) := \inf_E \left\{ \sum_{x_i \in E} e^{\int_0^T f(\phi_t x_i)dt} \, : \, N = \bigcup_{x_i \in E} B_T(x, \varepsilon) \right\},$$

where $B_T(x, \varepsilon)$ is the ball centered about x of radius ε in the d_T metric, and E is any set of points in N. Then we define the *topological pressure of ϕ_t with respect to f* to be

$$P_{\mathrm{top}}(\phi; f) := \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log S(\phi, f, T, \varepsilon).$$

The name 'pressure' comes from statistical mechanics; see [**You02**] for a detailed explanation of the history of this development. Similarly to Theorem 10.16, we have the *Variational Principle for Pressure*, which asserts the following.

THEOREM 10.20. *Let $\phi_t : N \to N$ be a flow on a closed manifold N, and let $f \in C^0(N, \mathbb{R})$. Then*

$$P_{\mathrm{top}}(\phi; f) = \sup_{m \in \mathcal{M}(\phi)} P_m(\phi; f).$$

An equilibrium state *is a measure m such that $P_{\mathrm{top}}(\phi; f) = P_m(\phi; f)$. If ϕ_t is Anosov and f is Hölder, then there exists a unique equilibrium state m_f.*

For a proof, see [**KH95**, Theorem 20.2.4, Corollary 20.3.8]. Theorem 10.16 is just the special case $f = 0$, as is easily checked. Two different functions may generate the same equilibrium state. In fact, the following holds.

EXERCISE 10.21. Suppose f and h are (flow) cohomologous. Then $m_f = m_h$.

Under the additional Anosov assumption the converse to the preceding exercise holds.

PROPOSITION 10.22. *Suppose $\phi_t : N \to N$ is an Anosov flow on a closed manifold N. Suppose $f, h \in C^0(N, \mathbb{R})$ are Hölder continuous and satisfy $m_f = m_h$. Then, there is a constant c such that f is cohomologous to $h + c$. Moreover the coboundary is the derivative along the flow of a Hölder continuous function.*

For a proof, see [**KH95**, Proposition 20.3.10]. The SRB measure for an Anosov flow is then obtained as the equilibrium state with respect to a special choice of potential.

DEFINITION 10.23. Let $\phi_t : N \to N$ be an Anosov flow on a closed manifold. Fix a Riemannian metric g on N, and put

$$J_t(x) := \det(d_x\phi_t)$$

$$J_t^s(x) := \det(d_x\phi_t|_{E^s(x)});$$

$$J_t^u(x) := \det(d_x\phi_t|_{E^u(x)}).$$

We call J_t^s and J_t^u the *stable* and *unstable Jacobians* of ϕ_t. Write

$$j(x) := \frac{d}{dt}\Big|_{t=0}\log J_t(x);$$

$$j^s(x) := \frac{d}{dt}\Big|_{t=0}\log J_t^s(x);$$

$$j^u(x) := \frac{d}{dt}\Big|_{t=0}\log J_t^u(x).$$

Strictly speaking, all these functions depend on the choice of metric g. However if g' is another choice of metric and j' the corresponding function then j and j' differ only by a coboundary (with similar statements for j^s and j^u). This can be easily seen as follows. By definition, $\phi_t^*\omega = J_t\omega$, where ω is the volume form on N induced by g and thus

$$L_F\omega = \frac{\partial}{\partial t}\Big|_{t=0}\phi_t^*\omega = j \cdot \omega,$$

which in turn implies

$$\operatorname{div}_\omega F = j. \tag{10.3.1}$$

But we have already seen (cf. Lemma 10.2) that two divergences differ by a coboundary. Since we will be interested only in expressions involving the integrals of j and $j^{s,u}$ over N, it therefore does not matter which metric we choose (of course, we are implicitly assuming here that both metrics are at least Hölder continuous).

Because of this, it is often convenient to choose the adjusted metric from Lemma 1.4. With this choice of metric we have

$$J_t(x) = J_t^s(x) \cdot J_t^u(x),$$

and hence

$$j(x) = j^s(x) + j^u(x).$$

We shall do this implicitly from now on whenever we speak of the (un)stable Jacobians.

DEFINITION 10.24. The *SRB measure* ρ is then defined to be the unique equilibrium state associated to $-j^u$, that is, $\rho = m_{-j^u}$. The function j^u is Hölder continuous; this follows from the fact that ([**KH95**, Theorem 19.1.6]) the stable and unstable subspaces are Hölder continuous. We shall see in the next subsection that $P_{\text{top}}(\phi, -j^u) = 0$.

REMARK 10.25. The SRB measure can be defined on much more general objects than Anosov systems (eg. *non-uniformly hyperbolic attractors*). In the most general situation possible, if $\phi_t : N \to N$ is a flow of class C^2 on a closed manifold N then a Borel probability measure μ is called an *SRB measure* if μ has a *positive Lyapunov exponent* almost everywhere, and μ has *absolutely continuous conditional measures on unstable manifolds*. We will not explain precisely what this definition means (although see the next section for the definition of Lyapunov exponents), and instead refer the reader to [**You02**, Section 2]. However without uniform hyperbolicity, there is no guarantee that an SRB measure may exist (although if it exists it is unique). The lecture notes [**You95**] or the survery article [**You02**] of Young discuss the question of when SRB measures exist in this more general setting.

The following theorem highlights the importance of the SRB measure. The theorem is essentially due to Sinai, Ruelle and Bowen; [**You02**] gives a detailed commentary on this result, together with copious references to the original proofs.

THEOREM 10.26. *Suppose m is an absolutely continuous measure. Define a measure m_T by*

$$m_T := \frac{1}{T}\int_0^T \phi_{t*} m dt.$$

Then m_T converges to the SRB measure ρ in the weak $*$-topology; that is for any $f \in C^0(N, \mathbb{R})$, we have

$$\int_N f dm_T \to \int_N f d\rho$$

as $T \to \infty$.

In particular, if $(\phi_t)_* m = m$ for an absolutely continuous measure m then m is the SRB measure ρ.

As an immediate corollary of this theorem and Corollary 3.14 we obtain:

COROLLARY 10.27. *If $\phi_t : SM \to SM$ is the geodesic flow of a closed manifold M and ϕ_t is Anosov, then the Liouville measure μ is the SRB measure.*

To conclude this section, we note the following observation, which will prove to be very helpful later on.

LEMMA 10.28. *Let $\phi_t : N \to N$ a flow on a closed manifold N with infinitesimal generator F. Let $m \in \mathcal{M}(\phi)$ and $f \in C^0(N, \mathbb{R})$ a continuous function that is continuously differentiable along the flow. Then*

$$\int_N F(f) dm = 0.$$

PROOF. Since $(\phi_t)_* m = m$ we have

$$\begin{aligned}
\int_N F(f) dm &= \int_0^1 \int_N F(f) d(\phi_{t*} m) dt \\
&= \int_N \int_0^1 d_{\phi_t x} f(F(\phi_t x)) dt dm \\
&= \int_N \{f(\phi_t x) - f(x)\} dm \\
&= 0
\end{aligned}$$

using invariance. □

EXERCISE 10.29. Give an alternative proof of the previous lemma using *Birkhoff's ergodic theorem* (see for instance [**KH95**, Theorem 4.1.2]), which states that if $g \in L^1(N, \mathbb{R})$ then the function $\widehat{g}$ defined by

$$\widehat{g}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T g(\phi_t x) dt,$$

is well defined almost everywhere and satisfies

$$\int_N g dm = \int_N \widehat{g} dm.$$

10.4. Lyapunov exponents

In this section we define *Lyapunov exponents*. This gives us another way to express the importance of the SRB measure. A good reference for Lyapunov exponents is [**KH95**, Chapter S]; see section S.2 in particular. Another comprehensive reference is [**BP07**], or its earlier version [**BP02**].

DEFINITION 10.30. Let $\phi_t : N \to N$ denote a flow on a closed manifold N. Let g be any metric on N. If $x \in N$ and $v \in T_x N$, define

$$\chi^{\pm}(x, v) := \limsup_{t \to \pm\infty} \frac{1}{t} \log \|d_x \phi_t(v)\|.$$

EXERCISE 10.31. Prove the following facts (see [**BP07**, Proposition 1.3.1, Proposition 1.3.2] if you get stuck): for each $x \in N$, the functions $\chi^{\pm}(x, \cdot)$ assumes only finitely many values on $T_x N$, say

$$\chi_1^+(x) < \chi_2^+(x) < \cdots < \chi_{s^+(x)}^+(x), \quad s^+(x) \leq \dim N;$$

$$\chi_1^-(x) > \chi_2^-(x) > \cdots > \chi_{s^-(x)}^-(x), \quad s^-(x) \leq \dim N.$$

Define

$$\widetilde{E}_i^+(x) := \left\{ v \in T_x N \ : \ \chi^+(x, v) \leq \chi_i^+(x) \right\},$$

$$\widetilde{E}_i^-(x) := \{ v \in T_x N \ : \ \chi^-(x, v) \leq \chi_i^-(x) \}.$$

Then we have filtrations

$$\{0\} =: \widetilde{E}_0^+(x) \subsetneq \widetilde{E}_1^+(x) \subsetneq \cdots \subsetneq \widetilde{E}_{s^+(x)}^+(x) = T_x N;$$

$$T_x N = \widetilde{E}_1^-(x) \supsetneq \widetilde{E}_2^-(x) \supsetneq \cdots \supsetneq \widetilde{E}_{s^-(x)+1}^- = \{0\}.$$

Define

$$k_i^+(x) := \dim \widetilde{E}_i^+(x) - \dim \widetilde{E}_{i-1}^-(x);$$

$$k_i^-(x) := \dim \widetilde{E}_i^-(x) - \dim \widetilde{E}_{i+1}^-(x).$$

The numbers $\{\chi_i(x)\}$ are known as the *Lyapunov exponents* of ϕ_t at x.

DEFINITION 10.32. Fix an invariant measure $m \in \mathcal{M}(\phi)$. A point $x \in N$ is an *m-regular point* if the following holds:

(1) $s^+(x) = s^-(x) =: s(x)$ and there exists a decomposition

$$T_x N = \bigoplus_{i=1}^{s(x)} E_i(x),$$

such that

$$\widetilde{E}_i^+(x) = \bigoplus_{j=1}^{i} E_i(x), \quad \widetilde{E}_i^-(x) = \bigoplus_{j=i}^{s(x)} E_i(x)$$

(and hence $E_i(x) = \widetilde{E}_i^+(x) \cap \widetilde{E}_i^-(x)$, and

$$k_i^+(x) = k_i^-(x) =: k_i(x) = \dim E_i(x).$$

(2) $\chi_i^+(x) = -\chi_i^-(x) =: \chi_i(x)$ and for all $v \in E_i(x) \backslash \{0\}$,

$$\lim_{t \to \pm\infty} \log \|d_x \phi_t(v)\| = \chi_i(x),$$

with uniform convergence in $E_i(x) \cap S_x N$.

(3) The functions s, χ_i, k_i and the subspaces E_i depend measurably on x and are ϕ_t-invariant.

Let $L(m)$ denote the set of m-regular points.

Note that if $m \in \mathcal{M}(\phi)$ is ergodic then the functions $x \mapsto \chi_i(x)$ and $x \mapsto k_i(x) := \dim E_i(x)$ are constant m-a.e.

Given $x \in L(m)$, let

$$E^u(x) := \bigoplus_{\{i \,:\, \chi_i(x) > 0\}} E_i(x), \quad E^s(x) := \bigoplus_{\{j \,:\, \chi_j(x) < 0\}} E_j(x),$$

$$E^+(x) := \bigoplus_{\{i\,:\,\chi_i(x)>0\}} E_i(x), \quad E^-(x) := \bigoplus_{\{j\,:\,\chi_j(x)<0\}} E_j(x),$$

and put as before

$$J^u_t(x) := \det(d_x\phi_t|_{E^u(x)}), \quad J^s_t(x) := \det(d_x\phi_t|_{E^s(x)}).$$

A proof of the following lemma can be found in [**BP07**, Proposition 3.1.6].

LEMMA 10.33. *For all $x \in L(m)$, it holds that*

$$\lim_{t\to\infty} \frac{1}{t} \log J^u_t(x) = \mathscr{X}^+(x) := \sum_{\{i\,:\,\chi_i(x)>0\}} k_i(x)\chi_i(x);$$

$$\lim_{t\to\infty} \frac{1}{t} \log J^s_t(x) = \mathscr{X}^-(x) := \sum_{\{j\,:\,\chi_j(x)<0\}} k_j(x)\chi_j(x).$$

The following theorem is known as *Oseledec's Multiplicative Ergodic Theorem* ([**KH95**, Theorem S.2.9]).

THEOREM 10.34. *For any $m \in \mathcal{M}(\phi)$, the set $L(m)$ has measure one.*

Next, as above put:

$$j^u(x) := \frac{d}{dt}\Big|_{t=0} \log J^u_t(x), \quad j^s(x) := \frac{d}{dt}\Big|_{t=0} \log J^s_t(x).$$

COROLLARY 10.35. *Let $m \in \mathcal{M}(\phi)$. The*

$$\int_N \mathscr{X}^+ dm = \int_N j^u dm;$$

$$\int_N \mathscr{X}^- dm = \int_N j^s dm.$$

PROOF. Birkhoff's Ergodic theorem tells us that

$$\int_N \left[\lim_{t\to\infty} \frac{1}{t} \int_0^t j^u(\phi_t x) dt \right] dm = \int_N j^u dm,$$

and for all $x \in L(m)$, Lemma 10.33 tells us that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t j^u(\phi_t x) dt = \lim_{t\to\infty} \frac{1}{t} \log J^u_t(x) = \mathscr{X}(x).$$

Finally, Theorem 10.34 tells us that integrating over $L(m)$ is the same as integrating over N. The proof for j^s and $\mathscr{X}^-$ is similar. □

The next result is known as *Ruelle's inequality*; a proof of this may be found in [**KH95**, Theorem S.2.13].

THEOREM 10.36. *For any $m \in \mathcal{M}(\phi)$,*

$$h_m(\phi) \le \int_N \mathscr{X}^+ dm.$$

EXERCISE 10.37. Prove that if $\phi_t : N \to N$ is a flow on a closed manifold N and $h_{\text{top}}(\phi) > 0$ then there exists $m \in \mathcal{M}(\phi)$ with some of its Lyapunov exponents positive. Now suppose that dim $N = 3$. Prove that if $h_{\text{top}}(\phi) > 0$ then there exists $m \in \mathcal{M}(\phi)$ with non-zero Lyapunov exponents.

For a discussion as to why having non-zero Lyapunov exponents is a desirable property, see for instance [**BP07**].

The following theorem is due to Ledrappier and Young (see [**LY85**]), and gives yet another characterization of the SRB measure. It is valid in the most general setting in which SRB measures can sensibly be defined (see Remark 10.25)

THEOREM 10.38. *Let $\phi_t : N \to N$ be a C^2 flow on a closed manifold and $m \in \mathcal{M}(\phi)$ a measure admitting a positive Lyapunov exponent almost everywhere. Then*

$$h_m(\phi) \leq \int_N \mathcal{X}^+ dm,$$

and equality is attained if and only if m is the SRB measure ρ.

Thus we obtain the following formula, which is known as *Pesin's formula*:

$$h_\rho(\phi) = \int_N \mathcal{X}^+ d\rho. \tag{10.4.1}$$

We conclude this section with a lemma which will be useful in the next section.

LEMMA 10.39. *Let $\phi_t : N \to N$ be a flow on a closed manifold N. Let F denote the infinitesimal generator of ϕ_t and $m \in \mathcal{M}(\phi)$. Let ω denote an arbitrary volume form on N. Then*

$$\int_N \mathcal{X} dm = \int_N \operatorname{div}_\omega F dm$$

(in particular the right-hand side is independent of the choice of volume form).

PROOF. By (10.3.1), $\operatorname{div}_\omega F = j$ and $j = j^s + j^u$ for an adapted metric (which is merely measurable in this situation, although this doesn't matter). By integrating over N the result follows from Corollary 10.35. □

10.5. Entropy production

In this section we define entropy production, and state and outline the proof of an important result due to Ruelle on the positivity of entropy production. There is relatively little (textbook) material available on entropy production; an elementary introduction is given in [**Jos05**], and more advanced accounts are available in [**Gal04, JQQ04**]. The survey article [**Che02**] also contains a short section on entropy production. Ruelle's original papers [**Rue96, Rue97b, Rue97a**] are also very readable.

DEFINITION 10.40. Let $\phi_t : N \to N$ denote a flow on a closed manifold N with infinitesimal generator F. Let $m \in \mathcal{M}(\phi)$. Define the *entropy production of ϕ_t with respect to m*, written $e_m(\phi)$ by

$$e_m(\phi) := -\int_N \operatorname{div}_\omega F dm,$$

where ω is an arbitrary volume form on N.

Straight from Lemma 10.39 we obtain the following result ([**Rue96**, Lemma 1.1]):

COROLLARY 10.41. *For any $m \in \mathcal{M}(\phi)$ it holds that*

$$e_m(\phi) = -\int_N \mathcal{X} dm.$$

The following theorem is the main result of this section. It is also due to Ruelle (see [**Rue96**, Theorem 1.2]), and relates the SRB measure to entropy production.

THEOREM 10.42. *Let $\phi_t : N \to N$ be a flow on a closed manifold N, and suppose that ϕ_t admits an SRB measure ρ. Then $e_\rho(\phi) \geq 0$. If ϕ_t is Anosov then we have equality if and only if ϕ_t preserves a smooth volume form.*

PROOF. For any $m \in \mathcal{M}(\phi)$ we have by Ruelle's inequality that

$$h_m(\phi) \leq \int_N \mathcal{X}^+ dm.$$

Replacing ϕ_t by ϕ_{-t} and using Exercise 10.15 gives us

$$h_m(\phi) \leq -\int_N \mathcal{X}^- dm.$$

For the special case $m = \rho$, using Corollary 10.41 and Exercise 10.15 we find that

$$\begin{aligned} e_\rho(\phi) &= -\int_N \mathscr{X} d\rho \\ &= \left(h_\rho(\phi) - \int_N \mathscr{X}^+ d\rho \right) \\ &\quad - \left(h_\rho(\phi) - \int_N \left[\text{sum of positive Lyapunov exponents w.r.t. } \phi^{-1}\right] d\rho \right) \\ &\geq 0, \end{aligned}$$

where we used Theorem 10.38.

Suppose now $e_\rho(\phi) = 0$. We claim this means that ρ is also the SRB measure with respect to ϕ_{-t}. Indeed, $e_\rho(\phi) = 0$ implies that

$$\int_N \mathscr{X}^+ d\rho = -\int_N \mathscr{X}^- d\rho,$$

and thus by Corollary 10.35

$$h_\rho(\phi) = -\int_N \left[\text{sum of negative Lyapunov exponents w.r.t. } \phi^{-1}\right] d\rho = \int_N -j^s d\rho,$$

and hence by Theorem 10.20 both $-j^u$ and j^s generate the same equilibrium state, and thus by Proposition 10.22 we see that $-j^u$ and j^s are flow cohomologous, and thus there exists a Hölder continuous function $u : N \to \mathbb{R}$ such that

$$j^s + j^u = F(u).$$

But now if ω denotes a smooth volume form we know from (10.3.1) that $\operatorname{div}_\omega F = j$. Since j and $j^s + j^u$ coincide up to a coboundary $F(w)$, with w Hölder continuous, we deduce that

$$\begin{aligned} \operatorname{div}_\omega F &= j \\ &= j^s + j^u + F(w) \\ &= F(u + w). \end{aligned}$$

Finally Theorem 6.14 implies that there exists a smooth $v \in C^\infty(N, \mathbb{R})$ such that $\operatorname{div}_\omega F = F(v)$. □

Combining this result with Theorem 10.6 we have:

COROLLARY 10.43. *Let (M, g) denote a closed Riemannian surface. Let $\phi_t : SM \to SM$ denote a magnetic thermostat determined by the pair (h, θ). Suppose that ϕ_t is Anosov. Then $e_\rho(\phi) = 0$ if and only if θ is exact.*

The reader is referred to [**Gal04**] and the references within for an explanation of why positivity of $e_\rho(\phi)$ is of interest; it is important in non-equilibrium statistical mechanics. In the next subsection we will give one particular example, based on the work of L.-S.Young.

10.6. Hausdorff Dimension

In this section we define the *Hausdorff dimension* $\mathrm{HD}(m)$ of a probability measure, and state two theorems due to Young on how this invariant relates to entropy and other concepts discussed previously in this chapter. This material will not be used elsewhere. We begin by recalling the definition of the *Hausdorff dimension* of a metric space.

DEFINITION 10.44. Let X be a metric space and $k \geq 0$ a real number. Given a finite or countable covering $\{U_i\}$ of X, define its *k-weight* $w_k(\{U_i\})$ to be

$$w_k(\{U_i\}) := \sum_i (\operatorname{diam} U_i)^k,$$

where $0^0 := 1$. Given $\varepsilon > 0$ define $m_{k,\varepsilon}(X)$ by

$$m_{k,\varepsilon}(X) := \inf\{w_k(\{U_i\}) \,:\, \operatorname{diam} U_i < \varepsilon \text{ for all } i\}.$$

The infimum is taken over all such finite or countable coverings; if no such covering exists then the infimum is defined to be $+\infty$. Since $m_{k,\varepsilon}(X)$ is a non-increasing function of ε, we can define the *k-dimensional Hausdorff measure* of X to be

$$m_k(X) := C(k) \cdot \lim_{\varepsilon \to 0} m_{k,\varepsilon}(X) \in [0, \infty],$$

where $C(k)$ is a positive constant defined to ensure that

$$m_k([0,1]^k) = 1 \quad \text{for } k \in \mathbb{N}.$$

As the name suggests, m_k is indeed a measure on the Borel σ-algebra of X for any $k \geq 0$. Moreover for any metric space X there exists a unique $k_0 \in [0, \infty]$ such that

$$m_k(X) = \begin{cases} 0 & k < k_0 \\ +\infty & k > k_0. \end{cases}$$

This leads us to define:

DEFINITION 10.45. The *Hausdorff dimension* of a metric space X is the unique k_0 such that $m_k(X) = 0$ for $k < k_0$ and $m_k(X) = \infty$ for $k > k_0$. We write $\mathrm{HD}(X) = k_0$.

REMARK 10.46. The Hausdorff dimension is not necessarily an integer. Moreover, $m_{\mathrm{HD}(X)}(X)$ could be anything in $[0, \infty]$.

Finally we define the Hausdorff dimension of a measure.

DEFINITION 10.47. Let X be a metric space and m a Borel probability measure on X. Define the *Hausdorff dimension of the measure m* to be

$$\mathrm{HD}(m) := \inf \{\mathrm{HD}(B) \,:\, B \subseteq X \text{ a Borel set with } m(B) = 1\}.$$

We have the following (see [**You82**]):

LEMMA 10.48. *Let m be a Borel probability measure. Suppose for m-a.e. $x \in X$ it holds that*

$$\lim_{\varepsilon \to 0} \frac{\log m(B(x, \varepsilon))}{\log \varepsilon} = k,$$

where $B(x, \varepsilon)$ denote the ball about x of radius ε. Then $\mathrm{HD}(m) = k$.

If $\mathrm{HD}(m) \notin \mathbb{N} \cup \{0\}$ we say that m is a *fractal measure*.

The following theorem illustrates the link between Hausdorff dimension and entropy. It is due to Young [**You82**].

THEOREM 10.49. *Let $\phi_t : M \to M$ be a flow on a closed 3-manifold. Suppose $m \in \mathcal{M}(\phi)$ is ergodic. Let $\chi_1 \geq \chi_2$ denote the Lyapunov exponents of ϕ_t. Then*

$$\mathrm{HD}(m) = 1 + h_m(\phi) \left(\frac{1}{\chi_1} - \frac{1}{\chi_2} \right),$$

as long as $\chi_1, \chi_2 \neq 0$.

The next exercise relates the entropy production of the SRB measure to its Hausdorff dimension.

EXERCISE 10.50. Let $\phi_t : M \to M$ be an Anosov flow on a closed 3-manifold. Show that the Hausdorff dimension of ρ is given by

$$HD(\rho) = 2 + \left(1 + \frac{e_\rho(\phi)}{h_\rho(\phi)} \right)^{-1} \in (2, 3].$$

In particular, if $e_\rho(\phi) > 0$ then ρ is a fractal measure. Use Corollary 10.43 to conclude that an Anosov magnetic thermostat with θ non-exact has a fractal SRB measure.

10.7. The relation between the SRB potential and the function r^+

In this final section we now return to the case of a closed surface (M, g), and $\phi_t : SM \to SM$ an Anosov λ-geodesic flow. Recall we have unique functions $r^{\pm}$ (see Definition 8.17) such that

$$H + r^{\pm} V \in E^{\pm}.$$

We aim to use the material we have developed in this chapter to obtain an explicit description of the function r^+.

Since $E^- = \mathbb{R}F \oplus E^s$ and $E^+ = \mathbb{R}F \oplus E^u$, this implies there exist unique functions $w^{\pm}$ such that

$$w^- F + H + r^- V \in E^s,$$
$$w^+ F + H + r^+ V \in E^u.$$

LEMMA 10.51. *Fix $(x_0, v_0) \in SM$, and let $w = w^{\pm}$, $r = r^{\pm}$ (with the same sign each time), and define*

$$\eta(t) = d\phi_{-t}\{w(t)F(t) + H(t) + r(t)V(t)\} \in E^{s,u}(0),$$

where as before $H(t) = H(\phi_t(x_0, v_0))$ and $H = H(0) = H(x_0, v_0)$ etc., and $\eta(t) \in E^s(0)$ if we have chosen the '$+$' sign and $\eta(t) \in E^u(0)$ if we have chosen the '$-$' sign. Then

$$\dot\eta(0) = -rwF - rH - r^2V.$$

PROOF. We compute

$$\begin{aligned} \dot\eta(0) &= \dot wF + w\frac{d}{dt}\Big|_{t=0} d\phi_{-t}(F(t)) + \frac{d}{dt}\Big|_{t=0} d\phi_{-t}(H(t) + rV(t)) \\ &= \dot wF + w[F, F] + [F, H] + r[F, V] + F(r)V \\ &= \dot wF - \lambda F + (K - H(\lambda) + \lambda^2)V - rH - rV(\lambda)V + F(r)V \\ &= \dot wF - \lambda F - rH + (F(r) + K - H(\lambda) + \lambda^2 - V(\lambda)r)V, \\ &= \dot wF - \lambda F - rH - r^2V, \end{aligned} \tag{10.7.1}$$

by the Riccati equation (8.4.1), together with (8.1.1) and (8.4.5).

Since $\dim E^{s,u} = 1$, there exists $a \in \mathbb{R}$ such that

$$\dot\eta(0) = a\eta(0),$$

and thus

$$a\eta(0) = awF + aH + arV = \dot wF - \lambda F - rH - r^2V,$$

and then since $\{F, H, V\}$ is a basis of $T_{(x_0,v_0)}SM$ by equating coefficients we see

$$a = -r.$$

Thus

$$\dot w + rw = \lambda,$$

and so substituting into (10.7.1) we obtain

$$\dot\eta(0) = -rwF - rH - r^2V$$

as required. □

Consider a Hölder continuous Riemannian metric on SM for which

$$\{F,\ w^-F + H + r^-V,\ w^+F + H + r^+V\}$$

is an orthonormal basis. In the statement below j^u is considered with respect to this metric.

COROLLARY 10.52. *Let $\phi_t : SM \to SM$ be an Anosov λ-geodesic flow on a closed surface M. Then the function r^+ is equal to j^u.*

PROOF. We now specialize Lemma 10.51 to the case of E^u, although for notational simplicity we will still write r instead of r^+ etc. Set

$$J(t) := J^u_t(x_0, v_0).$$

Then

$$j^u(x_0, v_0) = \frac{d}{dt}\Big|_{t=0} \log J(t) = \dot J(0),$$

since $J(0) = 1$. By definition,

$$d\phi_t(wF + H + rV) = J(t) \cdot \{w(t)F(t) + H(t) + r(t)V(t)\} \in E^u(t),$$

and hence

$$wF + H + rV = J(t)\eta(t),$$

and so

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0}(wF + H + rV) \\ &= \frac{d}{dt}\Big|_{t=0} J(t)\eta(t) \\ &= \dot J(0)\eta(0) + J(0)\dot\eta(0) \\ &= \dot J(0)(wF + H + rV) + (-rwF - rH - r^2V), \end{aligned}$$

by the previous lemma. Thus $\dot{J}(0) = r = r^+$, and hence $j^u = r^+$ as claimed. □

Thus we have obtained an explicit description of r^+, completing the goal we set out to achieve in this chapter.

As corollary we obtain:

COROLLARY 10.53. *Let $\phi_t : SM \to SM$ be an Anosov λ-geodesic flow on a closed surface M. Then*

$$h_\rho(\phi) = \int_{SM} r^+ \, d\rho.$$

In particular:

COROLLARY 10.54. *Let (M, g_0) be a closed surface of constant curvature $K < 0$. Then the metric entropy of the Liouville measure is $\sqrt{-K}$,*

$$h_\mu(\phi) = \sqrt{-K},$$

where $\phi_t : SM \to SM$ is the geodesic flow.

PROOF. By Corollary 10.27, the Liouville measure is the SRB measure. The Riccati equation (8.4.1) reduces to

$$X(r) + r^2 + K = \dot{r} + r^2 + K = 0,$$

and thus $r^+(x) \equiv \sqrt{-K}$. The corollary now follows from Corollary 10.53 since μ is a probability measure. □

EXERCISE 10.55. Prove that if (M, g_0) is a surface of constant negative curvature and $\phi_t : SM \to SM$ the geodesic flow then there exists a unique measure of maximal entropy in $\mathcal{M}(\phi)$, and this is precisely the Liouville measure μ.

Hence

$$h_{top}(\phi) = h_\mu(\phi),$$

meanwhile for any other $m \in \mathcal{M}(\phi)$ it holds that

$$h_{top}(\phi) > h_m(\phi).$$

A remarkable theorem of Katok [**Kat82**] asserts that the only surfaces for which the Liouville measure is the measure of maximal entropy are those with constant curvature.

EXERCISE 10.56. Show that for an Anosov λ-geodesic flow it holds

$$e_\rho(\phi) = -\int_{SM} (r^- + r^+) \, d\rho.$$

CHAPTER 11

Regularity of the (un)stable bundles

In this chapter we give another application of Theorem 7.15, and focus on determining the regularity of the strong and weak stable and unstable bundles.

11.1. Hölder continuity of the distributions E^s, E^u, E^- and E^+

Let $\phi_t : N \to N$ be an Anosov flow on a closed manifold N, and let F denote the infinitesimal generator of ϕ_t. Then we can write

$$TN = \mathbb{R}F \oplus E^s \oplus E^u.$$

Recall we say E^s and E^u are the *strong* stable and unstable bundles, and $E^- = \mathbb{R}F \oplus E^s$ and $E^+ = \mathbb{R}F \oplus E^u$ the *weak* stable and unstable bundles (see Definition 8.1).

EXERCISE 11.1. Show that both E^s and E^u are continuous.

Anosov proved in [**Ano67**] that the strong bundles E^s and E^u are Hölder continuous.

THEOREM 11.2. *Let $\phi_t : N \to N$ be an Anosov flow on a closed manifold N. Then the distributions E^s, E^u, E^- and E^+ are all Hölder continuous.*

The proof can be found in many books on hyperbolic dynamical systems; see for instance [**KH95**, Theorem 19.1.6], [**BS02**, Theorem 6.1.3] (technically both these proofs are actually only for the case of Anosov diffeomorphisms; for a proof for an Anosov flow see [**Bal95**, Appendix, Proposition 4.4]).

REMARK 11.3. What does it actually mean for a distribution to be Hölder continuous? If $N \subseteq \mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ then the definition of a Hölder continuous distribution $E \subseteq TN$ is the obvious one: namely $E \subseteq TN$ is α-Hölder continuous for some $\alpha \in (0,1)$ if there exists a constant $C \geq 0$ such that

$$\operatorname{dist}(E(x), E(y)) \leq C \, \|x - y\|^\alpha \quad \text{for all } x, y \in M \text{ with } \|x - y\| \leq 1.$$

Here $\|\cdot\|$ is the standard Euclidean norm and given subspaces $U, V \subseteq \mathbb{R}^n$,

$$\operatorname{dist}(U, V) := \max \left\{ \max_{u \in U, \|u\|=1} \min_{v \in V} \|u - v\|, \max_{v \in V, \|v\|=1} \min_{u \in U} \|u - v\| \right\}.$$

For an arbitrary closed Riemannian manifold (N, g), we can use the same definition by first embedding N into $\mathbb{R}^n$. Compactness of N implies that the Hölder exponent α is independent of the choice of embedding (although the constant C does change).

Suppose now N is a 3-manifold. Then the weak bundles $E^\pm$ are of *codimension* 1, and this is enough to force them to have more regularity. In fact, Hirsch, Pugh and Shub [**HPS77**] proved that in this case, the weak bundles are C^1.

THEOREM 11.4. *Let $\phi_t : N \to N$ be an Anosov flow on a closed 3-manifold N. Then the weak bundles $E^\pm$ are of class C^1.*

The proof can also be found in [**KH95**, Corollary 19.1.12].

11.2. The bundle S

In this subsection we restrict ourselves to the case of a transitive Anosov flow $\phi_t : N \to N$ on a closed 3-manifold N. Let us also consider the bundle

$$S = E^s \oplus E^u.$$

We know that this is Hölder continuous; we can actually say a lot more in the presence of an extra geometric structure.

DEFINITION 11.5. We say that the flow ϕ_t is a *contact flow* if there exists a contact 1-form α such that $\alpha(F) = 1$ and $i_F d\alpha = 0$. (That is, F is the Reeb vector field of α.)

If $\phi_t : SM \to SM$ is the geodesic flow on a closed surface then ϕ_t is contact, with α the 1-form defined in Definition 1.23.

LEMMA 11.6. *Let $\phi_t : N \to N$ be an Anosov flow on a closed 3-manifold N. Suppose ϕ_t is contact with contact form α. Then the bundle S has the same regularity as α; in particular if α is C^∞ then so is S.*

PROOF. It is enough to prove that

$$S = \ker \alpha. \tag{11.2.1}$$

To prove this, observe that since α is invariant (as $L_F\alpha = d(i_F\alpha) + i_F(d\alpha) = d(\alpha(F)) + 0 = d(1) = 0$), if $v \in E^s(x)$ then

$$\alpha_x(v) = \alpha_{\phi_t x}(d\phi_t(v))$$

for any $t \geq 0$. But since

$$|d\phi_t(v)| \leq Ce^{-\mu t}\,|v|$$

for some constants $C, \mu > 0$, and since there exists $A > 0$ such that

$$\|\alpha_x\| \leq A$$

(using compactness of N), we see that

$$|\alpha_x(v)| \leq ACe^{-\mu t}\,|v| \quad \text{for all } t > 0,$$

and thus $E^s \subseteq \ker \alpha$. A similar argument, considering $t < 0$ shows that $E^u \subseteq \ker \alpha$. Hence $S \subseteq \ker \alpha$. Since both sides of (11.2.1) have the same dimension, this is enough to prove (11.2.1). ☐

COROLLARY 11.7. *Let $\phi_t : N \to N$ be an Anosov flow on a closed 3-manifold N. Suppose ϕ_t is contact with a C^1 contact form α. Then the strong stable and unstable bundles E^s and E^u are of class C^1.*

PROOF. Since $E^s = S \cap E^-$, and both S and E^- are of class C^1 (by Theorem 11.4 and Lemma 11.6) so is E^s. Similarly E^u is of class C^1. ☐

It is thus of interest to know when ϕ_t is contact. From (11.2.1), we can make a guess as to what a contact form should be for ϕ_t (if it exists). We are led to consider the 1-form τ on N defined by

$$\tau_x(v) := \begin{cases} 0 & v \in S(x) \\ 1 & v = F(x). \end{cases}$$

In general we only know that τ is Hölder continuous. However if we know that τ is C^1 then a lot more is true, as the following theorem due to Hurder and Katok ([**HK90**, Theorem 2.3]) says:

THEOREM 11.8. *Assume ϕ_t is transitive and suppose τ is of class C^1. Then actually τ is of class C^∞, and if $d\tau$ is not identically zero then $\tau \wedge d\tau$ is a smooth volume form, and thus ϕ_t is contact with contact form τ.*

PROOF. First, we observe that the continuous 3-form $\tau \wedge d\tau$ is a ϕ_t-invariant 3-form since τ and $d\tau$ are. Set

$$A := \{x \in N : (\tau \wedge d\tau)_x = 0\}.$$

If A has non empty interior then $A = N$, since ϕ_t is transitive and A is invariant. Otherwise A is nowhere dense. Let g denote a metric on N and ω_g the associated volume form. Let ν denote the Riemannian measure of (N, g), that is the measure defined by

$$\nu(B) = \frac{\int_B \omega_g}{\mathrm{Vol}(N, g)}, \quad B \subseteq N \text{ a Borel set.}$$

Then [**Pla72**, Theorem 4.1] tells us that $\nu(A) = 0$. Let us now consider the two cases:

Suppose $\tau \wedge d\tau = 0$. Then

$$0 = i_F(\tau \wedge d\tau) = \tau(F)d\tau + \tau \wedge i_F d\tau.$$

But $i_F d\tau = L_F\tau - di_F\tau = 0$ and hence $d\tau = 0$. Now we show that τ is C^∞. Since τ is of class C^1 and closed, by the de Rham theorem we can find a harmonic 1-form β and a C^1 function h such that $\tau = \beta + dh$. Applying this to F we derive $1 = \beta(F) + X(h)$. The Livsic cocycle regularity theorem 6.11 implies that h is C^∞ and thus τ is C^∞.

Now suppose $\tau \wedge d\tau$ is not identically zero. We first claim in this case that ϕ_t preserves a smooth volume form. Indeed, let ω be any choice of volume form. Then there exists a continuous function f such that

$$\tau \wedge d\tau = f\omega.$$

Moreover f is differentiable in the direction of the flow and $f \neq 0$ on a set of full measure (namely, A). Since $L_F(f\omega) = 0$, we have

$$F(f) + f\,\mathrm{div}_\omega F = 0.$$

Consider the measurable function u defined as $-\log f$ at points where f is positive and as $\log(-f)$ at points where f is negative. Then, almost everywhere

$$\mathrm{div}_\omega F = F(u).$$

But now, Theorem 6.19 (this is the only place we use this theorem in these notes) implies that there exists a Hölder continuous function w such that $F(w) = \mathrm{div}_\omega F$, and then Theorem 6.11 implies that w is actually smooth. Hence ϕ_t preserves a smooth volume form ω'. Now apply the following exercise:

EXERCISE 11.9. Suppose $\phi_t : N \to N$ is a transitive flow and ω_1 and ω_2 are two ϕ_t-invariant volume forms. Prove there exists a non-zero constant $c \in \mathbb{R}$ such that $\omega_1 = c\omega_2$.

Thus we can write

$$\tau \wedge d\tau = c\omega'$$

for some non-zero constant c. In particular, $\tau \wedge d\tau$ never vanishes. We now show that τ is in fact of class C^∞. Contract with F on both sides to obtain

$$d\tau - \tau \wedge i_F d\tau = c i_F \omega',$$

and thus as $i_F d\tau = 0$ we have

$$\frac{1}{c} d\tau = i_F \omega',$$

and the form $i_F\omega'$ is exact. Thus we can write $i_F\omega' = d\eta$ for some smooth 1-form η, and hence $\tau - c\eta$ is closed. Then by the de Rham theorem we can write

$$\tau - c\eta = \beta + dh$$

where β is harmonic and h is of class C^1. Finally evaluating both sides on F we get

$$1 - c\eta(F) = \beta(F) + F(h),$$

and Theorem 6.11 tells us that h is actually of class C^∞. But this implies that $\tau = c\eta + \beta + dh$ is of class C^∞, and this completes the proof. □

COROLLARY 11.10. *Let M be a closed surface and $\phi_t : SM \to SM$ an Anosov magnetic thermostat ϕ_t determined by the pair (h, θ). If θ is not exact then S is not C^1.*

PROOF. We prove the contrapositive; namely that if τ is C^1 then θ is exact. First let us check that we cannot have $d\tau = 0$. Indeed, recall from Corollary 8.10 and Corollary 8.12 that the zero class $[0] \in H_1(SM, \mathbb{R})$ contains a closed orbit Γ of ϕ_t say, with period $T > 0$. Then if $(x_0, v_0) \in \Gamma$,

$$\int_\Gamma \tau = \int_0^T \tau(F(\phi_t(x_0, v_0)))dt = T \neq 0.$$

But since $[\Gamma] = [0]$, we have $\Gamma = \partial D$ for some 2-chain D. Then if $d\tau = 0$, by Stokes' theorem we have the contradiction

$$\int_\Gamma \tau = \int_D d\tau = 0.$$

Thus we must be in the contact case. But then ϕ_t preserves a volume form (namely $\tau \wedge d\tau$) and hence Theorem 10.6 completes the proof. □

Suppose now that $\phi_t : SM \to SM$ is a magnetic flow, so that $\lambda(x, v) = f(x)$ for some $f \in C^\infty(M, \mathbb{R})$. Suppose that ϕ_t is Anosov and the 1-form τ is of class C^1. We investigate the consequences.

In other words, ϕ_t is the flow of the unique vector field F such that $i_F\omega_\sigma = dH$, where $H(x, v) = \frac{1}{2}|v|^2$, $\omega_\sigma = -d\alpha + \pi^*\sigma$ and $\sigma = f\Omega_a$ (see Lemma 7.7).

EXERCISE 11.11. Show that there exists a non-zero constant $c \in \mathbb{R}$ such that $d\tau = c\omega_\sigma$.

EXERCISE 11.12. Show that $H^2(M,\mathbb{R})$ is generated by the Euler class $e = [K\Omega_a]$, and hence that there exists a constant κ and a 1-form η such that

$$\sigma = \kappa K\Omega_a + d\eta.$$

Recall the connection 1-form ψ from Definition 4.5. We have

$$\begin{aligned} \pi^*\sigma &= \kappa\pi^*(K\Omega_a) + \pi^*(d\eta) \\ &= -\kappa d\psi + \pi^*(d\eta) \\ &= d(-\kappa\psi + \pi^*\eta), \end{aligned}$$

where we are using the fact that $d\psi = -K\alpha \wedge \beta = -K\pi^*\Omega_a$. In other words,

$$\omega_\sigma = d(-\alpha - \kappa\psi + \pi^*\eta),$$

and hence ω_σ is exact. Moreover

$$d\tau = cd(-\alpha - \kappa\psi + \pi^*\eta),$$

and thus we have

$$d\chi = 0,$$

where

$$\chi = \tau + c\alpha + c\kappa\psi - c\pi^*\eta,$$

that is, χ is a closed 1-form on SM. Since $\pi^* : H^1(M,\mathbb{R}) \to H^1(SM,\mathbb{R})$ is an isomorphism by Corollary 8.10, it follows there exists a closed 1-form ρ on M and a C^1 function $u : SM \to \mathbb{R}$ such that

$$\chi = \pi^*\rho + du.$$

We now apply this to F: since $\tau(F) = \alpha(F) = 1$, $\psi(F) = f$ and

$$\pi^*\eta(F)(x,v) = \eta(d\pi(F(x,v))) = \eta_x(v),$$

we obtain

$$1 + c + c\kappa f(x) - c\eta_x(v) = \rho_x(v) + du_{(x,v)}(F),$$

or alternatively

$$\begin{aligned} F(u)(x,v) &= du_{(x,v)}(F) \\ &= 1 + c + c\kappa f(x) - c\eta_x(v) - \rho_x(v) \\ &=: h(x) + \theta_x(v), \end{aligned}$$

where $h := 1 + c + c\kappa f$ is a function on M and $\theta := -c\eta - \rho$ is a 1-form.

We can now apply Theorem 7.15 to conclude that $h = 0$ and θ is exact. Indeed, if $\gamma \in \mathcal{G}_\lambda(M,g)$ then

$$I[h+\theta](\gamma) = I[F(u)](\gamma) = \int_\gamma F(u) = 0.$$

Since $c \neq 0$ and θ is exact, $d\eta = 0$. We now have two cases: if $\kappa = 0$ then $\sigma = \kappa K\Omega_a + d\eta = 0$, and hence $f = 0$.

If $\kappa \neq 0$ then f must be constant. Since $d\eta = 0$, we conclude

$$\begin{aligned} f\Omega_a &= \sigma \\ &= -\kappa K\Omega_a + d\eta \\ &= -\kappa K\Omega_a, \end{aligned}$$

which implies that K is constant.

Summarizing, we have proved:

THEOREM 11.13. *Let $\phi_t : SM \to SM$ be an Anosov magnetic flow on a closed surface M, with magnetic field $\sigma = f\Omega_a$. If the bundle S is C^1 then:*

(1) *the cohomology class $[\sigma] = 0$ implies $\sigma = 0$, that is, ϕ_t is just the geodesic flow.*
(2) *$[\sigma] \neq 0$ implies that f is constant and the metric has constant Gaussian curvature.*

EXERCISE 11.14. Relate the second case in the theorem above with an appropriate hyperbolic element in $\mathfrak{sl}(2,\mathbb{R})$ in Example 3.5.

This concludes our discussion of the bundle S; we now move on to discussing the regularity of the weak stable and unstable bundles themselves.

11.3. The weak bundles and the Godbillon-Vey class

In this section we look again at the weak bundles $E^{\pm}$. As before, let N be a closed 3-manifold and ϕ_t an Anosov flow on N, with weak bundles $E^{\pm}$. Then according to Theorem 11.4, $E^{\pm}$ are always of class C^1. We now ask:

Question: when are the weak bundles $E^{\pm}$ of class C^2?

DEFINITION 11.15. Let N be a closed 3-manifold and $E \subset TN$ a C^2 subbundle of codimension 1. We say that E is *transversely orientable* if there exists a vector field Z on N such that $Z(x) \notin E(x)$ for all $x \in N$.

Suppose now $E \subset TN$ is a transversely orientable subbundle which in addition is *integrable* (in the sense of Frobenius). Define a 1-form τ by

$$\tau_x(v) := \begin{cases} 0 & v \in E(x) \\ 1 & v = Z(x), \end{cases}$$

so $E = \ker \tau$ and τ is C^2.

We first claim there exists a C^1 form η such that $d\tau = \eta \wedge \tau$. To see this first note that if X, Y are vector fields belonging to E then $d\tau(X,Y) = 0$: indeed

$$d\tau(X,Y) = X\tau(Y) - Y\tau(X) - \tau([X,Y]) = 0,$$

as $[X,Y]$ belongs to E since E is integrable, and $E = \ker \tau$.

Suppose $d\tau = \eta \wedge \tau$ for a 1-form η; then

$$i_Z d\tau = \eta(Z)\tau - \tau(Z)\eta = \eta(Z)\tau - \eta.$$

It then follows that if we define η by

$$\eta = -i_Z d\tau \tag{11.3.1}$$

then $d\tau = \eta \wedge \tau$. Note that if η' is another 1-form with $d\tau = \eta' \wedge \tau$ then $\eta' = \eta + a\tau$ for some smooth function a.

The following numerical invariant is due to Godbillon and Vey in [**GV71**].

DEFINITION 11.16. Consider the C^0 form $\eta \wedge d\eta$, where η is as defined above. We can integrate $\eta \wedge d\eta$ over N, and we define the *Godbillon-Vey invariant* of E to be

$$\mathrm{gv}(E) := \int_N \eta \wedge d\eta.$$

Thurston gave examples of foliations $\mathcal{F}$ in S^3 such that $\mathrm{gv}(T\mathcal{F})$ takes all possible real values; see [**Thu74**].

LEMMA 11.17. *The Godbillon-Vey invariant is really an invariant of E: it is independent of the choice of Z, τ and η.*

PROOF. It is enough to check that $\mathrm{gv}(E)$ is invariant of η and τ, since τ and Z determine each other. Suppose τ is fixed but we choose a new η, that is, we replace η by $\eta' := \eta + a\tau$ for some smooth function a then

$$\begin{aligned} \eta' \wedge d\eta' &= (\eta + a\tau) \wedge (d\eta + da \wedge \tau + a\,d\tau) \\ &= \eta \wedge d\eta + \eta \wedge da \wedge \tau + \eta \wedge \eta \wedge \tau + a\tau \wedge d\eta \\ &\overset{(*)}{=} \eta \wedge d\eta - d(a\,d\tau), \end{aligned}$$

where $(*)$ used the fact that

$$\begin{aligned} \tau \wedge d\eta &= d(\tau \wedge \eta) - d\tau \wedge \eta \\ &= d(d\tau) - \eta \wedge \eta \wedge \tau \\ &= 0. \end{aligned}$$

Thus $\eta' \wedge d\eta' - \eta \wedge d\eta$ is exact, and hence by Stokes' theorem

$$\int_N \eta' \wedge d\eta' = \int_N \eta \wedge d\eta.$$

□

EXERCISE 11.18. Complete the proof by showing that if we change τ to $f\tau$ where $f \in C^\infty(N, \mathbb{R})$, $f > 0$ then the value of $\mathrm{gv}(E)$ does not change.

EXERCISE 11.19. Show that $\mathrm{gv}(E)$ is invariant under C^2 diffeomorphisms, that is, if $\phi : N' \to N$ is a C^2 diffeomorphism then

$$\mathrm{gv}(\phi^* E) = \mathrm{gv}(E).$$

REMARK 11.20. We have seen that we can associate a numerical invariant $\mathrm{gv}(E)$ to a transversely orientable integrable C^2 subbundle $E \subset TN$ of codimension one. Suppose we remove the condition that N is 3-dimensional. The construction of η goes through as before, only we can no longer integrate $\eta \wedge d\eta$ over N and thus cannot obtain a numerical invariant as before.

However all is not lost. From above, $\tau \wedge d\eta = 0$, and hence $d\eta = \tau \wedge \sigma$ for some form σ. Thus $\eta \wedge d\eta$ is a closed form on N and hence it represents a de Rham cohomology class, called the *Godbillon-Vey class*,

$$\mathrm{GV}(E) := [\eta \wedge d\eta] \in H^3(N, \mathbb{R}).$$

The following result is a corollary of the work of Ghys.

THEOREM 11.21. *Let M be a closed surface and $\phi_t : N \to N$ an Anosov λ-geodesic flow. Then if the bundles $E^\pm$ are C^2 then*

$$\mathrm{gv}(E^\pm) = 4\pi^2 \chi(M).$$

REMARK 11.22. The following is technical, and may be omitted without losing the main thread of this chapter. Here is what Ghys actually proves. Theorem 4.6 in [**Ghy93**] asserts that a smooth Anosov flow on a closed 3-manifold with weak stable and unstable foliations of class $C^{1,1}$ is smoothly orbit equivalent (up to finite covers) to a suspension or to what Ghys calls a *quasi-fuchsian flow* and which are described in Théorème B of [**Ghy92**] (in our case, since we are working with circles bundles the latter alternative holds.) A quasi-fuchsian flow ψ depends on a pair of points $([g_1], [g_2])$ in Teichmüller space, has smooth weak stable foliation C^∞-conjugate to the weak stable foliation of a constant curvature metric g_1 and smooth weak unstable foliation C^∞-conjugate to the weak unstable foliation of a constant curvature metric g_2. Moreover, ψ preserves a volume form if and only if $[g_1] = [g_2]$. These results of Ghys imply that if an Anosov λ-geodesic flow has weak stable and unstable foliations of class $C^{1,1}$ then the Godbillon-Vey invariant of the foliations must be equal to $4\pi^2\chi(M)$.

Returning to the setting we are interested in, we have the following result proved in [**Pat07**, Proposition 3.1]. It can be thought of as a natural generalization to λ-geodesic flows of an earlier result due to Mitsumatsu (see [**Mit85**], as well as [**HK90**, Proposition 9.1]), stated as Corollary 11.24 below.

THEOREM 11.23. *Let M be a closed surface and $\phi_t : SM \to SM$ an Anosov λ-geodesic flow. Let $E = E^\pm$ and $r = r^\pm$. Suppose E is of class C^2. Then*

$$\mathrm{gv}(E) = 4\pi^2\chi(M) - 3\int_{SM} \left\{[V(\lambda)]^2 + [V(r)]^2\right\} d\mu + 2\int_{SM} \left\{V(r)(V^2(\lambda) - 2\lambda)\right\} d\mu.$$

It will take us a while to prove this. Let us first note a corollary.

COROLLARY 11.24. *In the geodesic flow case we have*

$$\mathrm{gv}(E) = 4\pi^2\chi(M) - 3\int_{SM} [V(r)]^2\, d\mu,$$

where $X(r) + r^2 + K = 0$. In particular if $K \equiv -1$ then $r^\pm = \pm 1$ and so $V(r) = 0$ and

$$\mathrm{gv}(E) = 4\pi^2\chi(M).$$

Before proving Theorem 11.23 we need the following lemma, which should be thought of as an integrated version of Proposition 8.19.

LEMMA 11.25. *Let $r = r^\pm$. Then we have*

$$\int_{SM} \{(r - V(\lambda))^2 + \lambda^2\}\, d\mu = -4\pi^2\chi(M) + \int_{SM} [V(\lambda)]^2\, d\mu.$$

PROOF. Let us recall the equation (8.4.1) satisfied by r:

$$F(r) + r^2 + K - H(\lambda) + \lambda^2 - V(\lambda)r = 0.$$

Next by Lemma 9.2 and Lemma 9.3 we have

$$\int_{SM} F(r)d\mu = -\int_{SM} rV(\lambda)d\mu,$$

and using the fact that $\int_{SM} H(\cdot)d\mu = 0$ (again by Lemma 9.2) and (9.1.4) we obtain

$$-\int_{SM} rV(\lambda)d\mu + \int_{SM}(r - V(\lambda))rd\mu = \int_{S_xM} d\theta \int_M K d\Omega_a + \int_{SM}[V(\lambda)]^2 d\mu,$$

and then the Gauss-Bonnet theorem completes the proof. □

We now prove Theorem 11.23.

PROOF. *(of Theorem 11.23)*
Since we know $V \notin E$ by Theorem 8.2 we may take $Z = V$, and thus

$$\tau_{(x,v)}(\xi) \equiv \begin{cases} 0 & \xi \in E(x,v) \\ 1 & \xi = V(x,v). \end{cases}$$

Since $E = \operatorname{span}\{F, H + rV\}$ we may take for our choice of τ

$$\tau = -\lambda\alpha - r\beta + \psi.$$

Indeed,

$$\begin{aligned} \tau(V) &= -\lambda\alpha(V) - r\beta(V) + \psi(V) \\ &= \psi(V) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \tau(F) &= -\lambda\alpha(X + \lambda V) - r\beta(X + \lambda V) + \psi(X + \lambda V) \\ &= -\lambda\alpha(X) + \psi(\lambda V) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \tau(H + rV) &= -\lambda\alpha(H + rV) - r\beta(H + rV) + \psi(H + rV) \\ &= -r\beta(H) + \psi(rV) \\ &= 0. \end{aligned}$$

We then take the simplest possible choice of η given by (11.3.1); namely

$$\eta = -i_V d\tau.$$

Using the now familiar structure equations of the coframe $\{\alpha, \beta, \psi\}$ and the fact that

$$d\lambda = X(\lambda)\alpha + H(\lambda)\beta + V(\lambda)\psi,$$

$$dr = X(r)\alpha + H(r)\beta + V(r)\psi,$$

we compute

$$\begin{aligned} d\tau &= -d\lambda \wedge \alpha - \lambda d\alpha - dr \wedge \beta - rd\beta + d\psi \\ &= -d\lambda \wedge \alpha - \lambda\psi \wedge \beta - dr \wedge \beta + r\psi \wedge \alpha - K\alpha \wedge \beta \\ &= -H(\lambda)\beta \wedge \alpha - V(\lambda)\psi \wedge \alpha - \lambda\psi \wedge \beta - X(r)\alpha \wedge \beta \\ &\quad -V(r)\psi \wedge \beta + r\psi \wedge \alpha - K\alpha \wedge \beta \\ &= (H(\lambda) - X(r) - K)\alpha \wedge \beta + (r - V(\lambda))\psi \wedge \alpha - (\lambda + V(r))\psi \wedge \beta, \end{aligned}$$

and hence

$$-i_V d\tau = -(r - V(\lambda))\alpha + (\lambda + V(r))\beta.$$

We will omit the calculation of $\eta \wedge d\eta$; it is in the same vein as the above, and is tedious. We finally obtain

$$\eta \wedge d\eta = A\alpha \wedge \beta \wedge \psi,$$

where

$$\begin{aligned} A &:= -(\lambda + V(r))^2 - (r - V(\lambda))^2 + (r - V(\lambda)) \cdot V(\lambda + V(r)) \\ &\quad -(\lambda + V(r))V(r - V(\lambda)). \end{aligned}$$

Using the fact $\Theta = \alpha \wedge \beta \wedge \psi$ is a volume form giving rise to μ, we have

$$\mathrm{gv}(E) = \int_{SM} A d\mu.$$

We can simplify this by recalling that since $L_V \Theta = 0$ by Lemma 9.2 we have

$$\int_{SM} V(\cdot) d\mu = 0,$$

and thus using the fact that

$$V\left[(r - V(\lambda))(\lambda + V(r))\right] = V(r - V(\lambda))(\lambda + V(r)) + (r - V(\lambda))V(\lambda + V(r))$$

we can rewrite

$$\begin{aligned} A &= \left[-(\lambda + V(r))^2 - (r - V(\lambda))^2 - 2(\lambda + V(r)) \cdot V(r - V(\lambda))\right] \\ &\quad + V\left[(r - V(\lambda))(\lambda + V(r))\right] \end{aligned}$$

to obtain

$$\begin{aligned} \mathrm{gv}(E) &= \int_{SM} A d\mu \\ &= \int_{SM} \left[-(\lambda + V(r))^2 - (r - V(\lambda))^2 - 2(\lambda + V(r)) \cdot V(r - V(\lambda))\right] d\mu. \end{aligned}$$

By expanding $(\lambda + V(r))^2$ and combining terms we can rewrite this as

$$\mathrm{gv}(E) = \int_{SM} \left[-\lambda^2 - (r - V(\lambda))^2 - 3[V(r)]^2 - 4\lambda V(r) + 2V^2(\lambda)(\lambda + V(r))\right] d\mu.$$

Next, using the fact that

$$V(\lambda V(\lambda)) = [V(\lambda)]^2 + \lambda V^2(\lambda)$$

and that $\int_{SM} V(\cdot) d\mu = 0$ we have

$$\int_{SM} \lambda V^2(\lambda) d\mu = -\int_{SM} [V(\lambda)]^2 d\mu,$$

from which we obtain

$$\mathrm{gv}(E) = \int_{SM} \left[-\lambda^2 - (r - V(\lambda))^2 - 3[V(r)]^2 + 2V(r)(V^2(\lambda) - 2\lambda) - 2[V(\lambda)]^2\right] d\mu,$$

and the proof is the completed by Lemma 11.25. □

COROLLARY 11.26. *Let us specialize to the case of an Anosov magnetic thermostat determined by the pair (h, θ). Suppose E is of class C^2. Then*

$$\mathrm{gv}(E) = 4\pi^2 \chi(M) - 3 \int_{SM} [V(r + \theta)]^2 \, d\mu.$$

PROOF. We have

$$V(\lambda) = V(h + V(\theta)) = \theta_x(\mathtt{i}^2 v) = -\theta,$$

and hence

$$V^2(\lambda) - 2\lambda = -3V(\theta) - 2h.$$

Since $v \mapsto \mathtt{i}v$ preserves μ by Exercise 9.13, we have

$$\int_{SM} [V(\lambda)]^2 \, d\mu = \int_{SM} \theta^2 d\mu = \int_{SM} [V(\theta)]^2 \, d\mu,$$

and thus

$$\begin{aligned} \mathrm{gv}(E) &= 4\pi^2 \chi(M) - 3 \int_{SM} \left([V(r)]^2 + [V(\theta)]^2\right) d\mu - 6 \int_{SM} V(r) V(\theta) d\mu \\ &\quad -4 \int_{SM} h V(r) d\mu. \end{aligned}$$

But then

$$\begin{aligned} 0 &= \int_{SM} V(hr)d\mu \\ &= \int_{SM} hV(r)d\mu + \int_{SM} rV(h)d\mu \\ &= \int_{SM} hV(r)d\mu + 0, \end{aligned}$$

and thus obtain

$$\mathrm{gv}(E) = 4\pi^2\chi(M) - 3\int_{SM} [V(r+\theta)]^2\, d\mu,$$

thus completing the proof. □

REMARK 11.27. In fact Theorem 11.23 holds when E of class is $C^{1,\alpha}$ for any $\alpha > \frac{1}{2}$. Of course, for this to make sense one needs to have a definition of $\mathrm{gv}(E)$ in this case. The reader is referred to [**HK90**] for this extension of the definition of $\mathrm{gv}(E)$.

We conclude this chapter with one more result, illustrating the high degree of rigidity of Anosov magnetic thermostat flows on surfaces. We shall use this theorem in the next chapter when we return to the boundary rigidity problem.

THEOREM 11.28. *Let (M, g) be a closed surface and let $\phi_t : SM \to SM$ be an Anosov magnetic thermostat flow determined by the pair (h, θ), where θ has zero divergence. Then if the both the weak bundles E^- and E^+ are C^2, we must have $\theta \equiv 0$ and h constant; moreover this implies that the curvature K of M is also constant.*

REMARK 11.29. The assumption θ has zero divergence is really no restriction at all, and is made only to simplify the forthcoming proof. To remove this assumption, one needs to argue that $E^{\pm}$ are independent of time changes and that there is a time change for which the divergence of θ is zero. Also note that if $\theta \equiv 0$ and both h and K are constant, then we are in the homogeneous $PSL(2, \mathbb{R})$-case and the bundles $E^{\pm}$ are in fact real analytic.

PROOF. Since E is C^2, Theorem 11.21 implies that $\mathrm{gv}(E) = 4\pi^2\chi(M)$, and thus Corollary 11.26 implies that $V(r+\theta) = 0$. Note that this holds for both $r = r^-$ and r^+. Set $s^{\pm} = r^{\pm} + \theta$, which we can think of as smooth functions defined on M as $V(s^{\pm}) = 0$. Since $r^- \neq r^+$ for all $(x, v) \in SM$ we may assume that $r^+ > r^-$. Now since by (8.4.1)

$$\begin{aligned} F(r^-) + (r^-)^2 + K - H(\lambda) + \lambda^2 - V(\lambda)r^- = 0, \\ F(r^+) + (r^+)^2 + K - H(\lambda) + \lambda^2 - V(\lambda)r^+ = 0, \end{aligned}$$

we have

$$\begin{aligned} F(\log(r^+ - r^-)) &= V(\lambda) - (r^- + r^+), \\ &= -\theta - (s^+ + s^- - 2\theta) \\ &= \theta - (s^+ + s^-), \end{aligned}$$

and hence

$$F(\log(s^+ - s^-)) + s^+ + s^- = \theta.$$

But now note that if g is a C^1 function on M then $F(g \circ \pi)(x, v) = d_x g(v)$, and thus the last equation implies that $s^+ + s^- = 0$ and that θ is exact (with $\theta = d(\log 2s^+)$). Since θ has zero divergence and is exact, $\theta \equiv 0$. Thus $\log 2s^+$ is constant which in turn implies that r^- and r^+ are also both constant.

Since $\theta = 0$ the Riccati equation (8.4.1) reduces to

$$F(r) + r^2 + K - H(h) + h^2 = 0.$$

Since r is constant, $F(r) = 0$ and we conclude that $K - H(h) + h^2$ is constant. Applying V to both sides and noting $V(K) = V(h) = 0$ as both K and h are smooth functions on M we reduce to

$$\begin{aligned} 0 &= VH(h) \\ &= [V, H](h) - HV(h) \\ &= -X(h) \\ &= -dh. \end{aligned}$$

Thus h is constant, and so $H(h) = 0$ and thus K is constant. This completes the proof. □

Finally we mention that we can prove the following weaker result (which is actually all we will need in the next chapter) *without* using Ghys' theorem 11.21.

COROLLARY 11.30. *Let (M, g) be a closed surface and let $\phi_t : SM \to SM$ be an Anosov magnetic thermostat flow determined by the pair (h, θ), where θ has zero divergence. Suppose there is exists a C^2 orbit equivalence between ϕ_t and a geodesic flow $\phi_t^0 : S^0M \to S^0M$ determined by a metric of constant negative curvature -1. Then $\theta \equiv 0$ and h is constant; moreover this implies that the curvature K of M is also constant.*

PROOF. The existence of the hypothesized orbit equivalence implies that the weak bundles $E^{\pm}$ of ϕ_t are of class C^2 with $\mathrm{gv}(E^{\pm}) = 4\pi^2\chi(M)$. Then previous proof now applies word for word. □

CHAPTER 12

Returning to the boundary rigidity problem

Having developed all of this extra material we return to the boundary rigidity problem. The main goal of the first section of this chapter is to complete the proof of Theorem 2.8 from Chapter 2. Then in the second section we discuss applications of this material to spectral geometry, and prove an important result of Guillemin and Kazhdan.

12.1. Boundary rigidity

Before we get started on the proof of Theorem 2.8, we will prove that simple compact subdomains of $\mathbb{H}^2$ are boundary rigid. The proof uses Theorem 11.28 from the previous chapter.

THEOREM 12.1. *Let $(M, \partial M) \subset \mathbb{H}^2$ be a compact simple manifold with the hyperbolic metric g. Then $(M, \partial M)$ is boundary rigid, i.e., if g' is another simple metric on M such that $d = d'$ on $\partial M \times \partial M$ then there exists a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{Id}$ and $\psi^* g = g'$.*

PROOF. Take a fundamental domain containing M; that is, a discrete cocompact lattice $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ such that $\Sigma := \mathbb{H}^2/\Gamma$ is a closed Riemann surface with $M \subset \Sigma$.

Example 2.17 implies (after modifying g' by a diffeomorphism if necessary) that we may glue the metric g' into Σ; more precisely we have a well defined new metric g'' on Σ given by

$$g'' = \begin{cases} g' & M \\ g & \Sigma \backslash M. \end{cases}$$

Then Lemma 2.19 implies that (Σ, g) and (Σ, g'') have smoothly conjugate geodesic flows, and Corollary 2.22 gives $\mathrm{Vol}(\Sigma, g) = \mathrm{Vol}(\Sigma, g'')$. It follows that (Σ, g'') has C^∞ weak bundles (since (Σ, g) does) and then Theorem 11.30 implies that (Σ, g'') has constant curvature K. Then finally the Gauss-Bonnet theorem gives

$$\begin{aligned} \chi(\Sigma) &= \int_\Sigma K \cdot d\,\mathrm{vol}_{g''} \\ &= K\mathrm{Vol}(\Sigma, g'') \\ &= K\mathrm{Vol}(\Sigma, g) \\ &= -K \int_\Sigma (-1) \cdot d\,\mathrm{vol}_g \\ &= -K\chi(\Sigma), \end{aligned}$$

and since $\chi(\Sigma) \neq 0$ it follows that $K \equiv -1$.

In other words we have shown that after modifying g' by a diffeomorphism if necessary we may assume that g' has constant curvature -1 on M. Now to complete the proof we can employ the *(local) Cartan-Ambrose-Hicks theorem* (see [**Cha06**, p154]) to conclude that the map

$$\exp_x \circ (\exp'_x)^{-1}$$

with $x \in \partial M$ (where exp is the g-exponential map and $\exp'$ is the g'-exponential map) is an isometry; moreover this map restricts to ∂M as the identity. In other words, the desired map ψ may be defined as $\exp_x \circ (\exp')_x^{-1}$. □

We will now use the technical machinery we have developed in this course to prove Theorem 2.8. We first tackle the closed case; after this we will tackle the compact with boundary case (see Theorem 12.5, below). This proof is adapted from [**SU00**, Theorem 1.2].

THEOREM 12.2. *Let M be a closed surface of negative curvature, and β a symmetric 2-tensor such that $I[\beta] = 0$. Then β is potential.*

REMARK 12.3. The assumption that $\dim M = 2$ is not necessary, and similar methods will give the same result for M n-dimensional instead. However the assumption that M has negative curvature is used twice in the proof, and it is currently an open problem as to whether the same result will hold if we only assume that the geodesic flow of M is Anosov. In general, whilst the Anosov condition is weaker than the negative curvature hypotheses (consider the Donnay-Pugh example from Section 5.2), it is fairly easy to extend results proved about negative curvature to just the Anosov case. It appears that this is an exception to this rule. Another exception is Theorem 8.2.

PROOF. We know that the geodesic flow is Anosov (thanks to Theorem 1.6), and moreover, considering β as a function $SM \to \mathbb{R}$ defined by

$$\beta(x, v) := \beta_x(v, v),$$

since $I[\beta] = 0$, Theorem 6.11 gives the existence of a smooth map $u : SM \to \mathbb{R}$ such that $X(u) = \beta$, that is,

$$X(u)(x, v) = \beta_x(v, v), \quad \text{for all } (x, v) \in SM.$$

What we want to show is that u is a 1-form, say $u(x, v) = \delta_x(v)$. Then if $Z = \hat{g}^{-1}\delta$ is the vector field g-dual to δ we have

$$\begin{aligned} X(u)(x, v) &= \frac{d}{dt}\Big|_{t=0} \delta_{\gamma_{(x,v)}(t)}(\dot{\gamma}_{(x,v)}(t)) \\ &= \frac{d}{dt}\Big|_{t=0} \big\langle Z(\gamma_{(x,v)}(t)), \dot{\gamma}_{(x,v)}(t) \big\rangle \\ &= 2\,\langle \nabla_v Z(x), v\rangle \\ &= \beta_x(v, v) \end{aligned}$$

and thus β is a potential.

We need a criterion similar to the one we used in Lemma 6.22 for u to be a 1-form (up to a constant) in terms of V. We claim that the desired statement we need is for

$$\Psi := V^2(u) + u = \text{const}.$$

Indeed, if $u(x, v) = \delta_x(v) + c$ then

$$\begin{aligned} \Psi(x, v) &= V^2(u)(x, v) + u(x, v) \\ &= V(\delta_x)(\mathtt{i}v) + \delta_x(v) + c \\ &= \delta_x(\mathtt{i}^2 v) + \delta_x(v) + c \\ &= c. \end{aligned}$$

Conversely if $\Psi = c$ then solving the ODE on the circle $S_x M$ gives

$$u(x, v) = f_1(x)\cos\theta + f_2(x)\sin\theta + c,$$

which is a 1-form up to a constant.

Next we recall the integrated Pestov identity (9.3.1) from Corollary 9.9

$$2\int_{SM} H(u)\cdot VX(u)d\mu = \int_{SM} [X(u)]^2\, d\mu + \int_{SM} [H(u)]^2\, d\mu - \int_{SM} K\,[V(u)]^2\, d\mu$$

(this is simplified from (9.3.1) since we are taking $\lambda = 0$, i.e. the geodesic flow case).

The crux of the proof is the following innocent looking claim:

Claim: $VX(\Psi) = -2H(\Psi)$.

Assuming the claim for the time being, we complete the proof. We apply the identity above to Ψ to get, using the fact that $K < 0$,

$$\begin{aligned} 0 &\geq -\int_{SM} [2H(\Psi)]^2\, d\mu \\ &= \int_{SM} [X(\Psi)]^2\, d\mu + \int_{SM} [H(\Psi)]^2 - \int_{SM} K[V(\Psi)]^2 d\mu \geq 0. \end{aligned}$$

This implies that $X(\Psi) = V(\Psi) = H(\Psi) = 0$, and thus Ψ is constant, as required.

It remains therefore to prove the claim: note that we have not yet used the fact that β is a symmetric 2-tensor.

Proof of claim: To prove the claim we repeatedly apply V:

$$
\begin{aligned}
& X(u) = \beta \\
\Rightarrow\quad & VX(u) = V(\beta) \\
\Rightarrow\quad & XV(u) + H(u) = V(\beta) \\
\Rightarrow\quad & VXV(u) + VH(u) = V^2(\beta) \\
\Rightarrow\quad & XV^2(u) + HV(u) + HV(u) - X(u) = V^2(\beta) \\
\Rightarrow\quad & X(\Psi) + 2HV(u) = V^2(\beta) + 2X(u) \\
\Rightarrow\quad & VX(\Psi) + 2VHV(u) = V^3(\beta) + 2V(\beta) \\
\Rightarrow\quad & VX(\Psi) + 2\big(HV^2(u) - XV(u)\big) = V^3(\beta) + 2V(\beta) \\
\Rightarrow\quad & VX(\Psi) + 2\big(HV^2(u) + H(u) - VX(u)\big) = V^3(\beta) + 2V(\beta) \\
\Rightarrow\quad & VX(\Psi) + 2H(\Psi) = V^3(\beta) + 4V(\beta).
\end{aligned}
$$

The proof is then completed by showing that $V^3(\beta) + 4V(\beta) = 0$, and this is where we will finally use the fact that β is a symmetric 2-tensor. To prove this we write

$$
\begin{aligned}
V(\beta_x(v,v)) &= \beta_x(\mathrm{i}v, v) + \beta_x(v, \mathrm{i}v) \\
&= 2\beta_x(\mathrm{i}v, v),
\end{aligned}
$$

and thus

$$
\begin{aligned}
V^2(\beta_x(v,v)) &= 2\beta_x(\mathrm{i}^2 v, v) + 2\beta_x(\mathrm{i}v, \mathrm{i}v) \\
&= -2\beta_x(v,v) + 2\beta_x(\mathrm{i}v, \mathrm{i}v),
\end{aligned}
$$

and hence

$$
\begin{aligned}
V^3(\beta_x(v,v)) &= -4\beta_x(\mathrm{i}v, v) + 4\beta_x(\mathrm{i}^2 v, \mathrm{i}v) \\
&= -8\beta_x(\mathrm{i}v, v) \\
&= -4V(\beta_x(v,v)).
\end{aligned}
$$

The proof is complete. □

EXERCISE 12.4. Prove Theorem 12.2 just assuming that the geodesic flow is Anosov and $K \leq 0$.

Next we show how to adapt this proof to work in the case that M has boundary ∂M, thus finally completing the proof of Theorem 2.8. This proof is also adapted from [**SU00**, Theorem 1.1], although this result was originally proved by Pestov and Sharafutdinov in [**PS88**].

THEOREM 12.5. *Let* $(M, \partial M, g)$ *be* simple *and of negative curvature, and* β *a symmetric* 2*-tensor such that* $I[\beta] = 0$. *Then* β *is potential.*

PROOF. In this case the equation we need to recall is the integral Pestov identity for the case where M has boundary (Exercise 9.10),

$$
\begin{aligned}
2\int_{SM} H(u)\cdot VX(u)d\mu \;=\; & \int_{SM} [X(u)]^2\, d\mu + \int_{SM} [H(u)]^2\, d\mu - \int_{SM} K\,[V(u)]^2\, d\mu \\
& + \int_{\partial(SM)} H(u)\cdot V(u)\cdot i_X\Theta - \int_{\partial(SM)} X(u)\cdot V(u)\cdot i_H\Theta \\
& - \int_{\partial(SM)} X(u)\cdot H(u)\cdot i_V\Theta.
\end{aligned}
$$

The proof is similar to the previous theorem, however there are some important differences. The most important is that we can no longer use the Livsic Theorem 6.11 to obtain a smooth function $u : SM \to \mathbb{R}$ such that $X(u) = \beta$. Thus we need to define u directly. We proceed as follows.

Firstly, recall from Section 2.10 that given any $(x,v) \in SM$ there exists a unique geodesic $\gamma_{(x,v)}$ adapted to (x,v) and defined on an interval $[\tau_-(x,v), \tau_+(x,v)]$, where $\gamma_{(x,v)}(\tau_-(x,v)) \in \partial_-(SM)$ and $\gamma_{(x,v)}(\tau_+(x,v)) \in \partial_+(SM)$. Moreover the functions $\tau_\pm$ are smooth on $SM\backslash S(\partial M)$ by the simple condition.

Thus given $(x, v) \in SM$, define

$$u(x, v) := \int_{\tau_-(x,v)}^{0} \beta_{\gamma_{(x,v)}(t)}(\dot{\gamma}_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))dt.$$

We claim $u(x, v) = 0$ for $(x, v) \in \partial SM$; indeed, the assertion $u|_{-(SM)} = 0$ is simply the fact that $I[\beta] = 0$, and the assertion $u|_{\partial_+(SM)} = 0$ is immediate from the fact that clearly $\tau_+(x, v) = 0$ for $(x, v) \in \partial_+(SM)$. Moreover u is smooth where τ_- is smooth; this is true on $SM \backslash S(\partial M)$ by the simple condition.

Next, we claim that u satisfies

$$X(u)(x, v) = \beta_x(v, v), \quad \text{for all } (x, v) \in SM \backslash S(\partial M).$$

Indeed, fix $(x, v) \in SM \backslash S(\partial M)$, and let $\gamma_{(x,v)} : [\tau_-(x, v), \tau_+(x, v)] \to M$ be as above. For sufficiently small $s \in \mathbb{R}$, let

$$x_s := \gamma_{(x,v)}(s), \quad v_s := \dot{\gamma}_{(x,v)}(s),$$

so that

$$\gamma_{(x_s,v_s)}(t) = \gamma_{(x,v)}(t + s)$$

and

$$\tau_-(x_s, v_s) = \tau_-(x, v) - s.$$

Thus

$$\begin{aligned} u(\gamma_{(x,v)}(s), \dot{\gamma}_{(x,v)}(s)) &= u(x_s, v_s) \\ &= \int_{\tau_-(x_s,v_s)}^{0} \beta_{\gamma_{(x_s,v_s)}(t)}(\dot{\gamma}_{(x_s,v_s)}(t), \dot{\gamma}_{(x_s,v_s)}(t)dt \\ &= \int_{\tau_-(x,v)}^{s} \beta_{\gamma_{(x,v)}(t)}(\dot{\gamma}_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))dt. \end{aligned}$$

Differentiating this relation with respect to s and then setting $s = 0$ we obtain

$$X(u)(x, v) = \beta_x(v, v)$$

as claimed.

Thus we have shown u depends smoothly on (x, v) apart from possibly points of $S(\partial M)$ where some derivatives of u could very well become infinite. Below we shall integrate u over all of SM; this means that some of the integrals we consider may be improper, and thus one should really prove convergence. We shall omit these checks, and refer the reader to [**Sha94**, Section 4.6] for information on how this is done.

Having done all of this the proof can essentially proceed as before, since we claim that with $\Psi = V^2(u) + u$ defined as before we have

$$\int_{\partial(SM)} H(\Psi) \cdot V(\Psi) \cdot i_X \Theta - \int_{\partial(SM)} X(\Psi) \cdot V(\Psi) \cdot i_H \Theta - \int_{\partial(SM)} X(\Psi) \cdot H(\Psi) \cdot i_V \Theta = 0. \tag{12.1.1}$$

Indeed, having shown (12.1.1), modulo the convergence issues that we omit, exactly the same method as in 12.2 works, and the proof is complete.

To show (12.1.1) we use that fact that $V(\Psi) = 0$ on $\partial(SM)$; this follows as $u = 0$ on $\partial(SM)$. This kills all but the last term, and the last term dies as by (4.1.8),

$$i_V \Theta = \alpha \wedge \beta = \pi^* \Omega_a,$$

and thus the restriction of $i_V \Theta$ to $\partial(SM)$ is zero. □

12.2. Applications to spectral geometry

We conclude with an application to infinitesimal spectral rigidity. In order to state this result we require some elementary definitions from spectral geometry.

DEFINITION 12.6. Let (M, g) be a closed Riemannian manifold whose geodesic flow is Anosov. We define 3 types of spectrum:

(1) The *spectrum of the Laplacian*, written $\text{Spec}(\Delta_g)$ is the sequence of eigenvalues (counted with multiplicites) of the Laplacian Δ_g acting on $C^\infty(M, \mathbb{R})$.
(2) The *marked length spectrum* is the function $L_g : \Pi(M) \to \mathbb{R}$, where $\Pi(M)$ is the set of non-trivial free homotopy classes, and for $\pi \in \Pi(M)$, $L_g(\pi) = \ell_g(\gamma_\pi)$, the length of the unique closed geodesic $\gamma_\pi \in \pi$.

(3) The *length spectrum*, written $\mathrm{LSpec}(g)$ is defined to be the image of L_g, that is,

$$\mathrm{LSpec}(g) = \{\ell(\gamma) : \gamma \in \mathcal{G}(M, g)\} \subseteq \mathbb{R}.$$

Note that $\mathrm{LSpec}(g)$ is a countable set.

THEOREM 12.7. *Let (M, g) be a closed negatively curved surface, and let $\{g_s : s \in (-\varepsilon, \varepsilon)\}$ be a smooth 1-parameter family of metrics. Suppose that*

$$Spec(\Delta_{g_s}) = Spec(\Delta_{g_0}) \text{ for all } s \in (-\varepsilon, \varepsilon). \tag{12.2.1}$$

Then there exists a family $\psi_s : M \to M$ of diffeomorphisms such that $\psi_0 = \mathrm{Id}$ and such that

$$g_s = \psi_s^* g_0.$$

As stated this theorem was first proved by Guillemin and Kazhdan in [**GK80**]. We shall sketch their original proof below in Remark 12.10, but for now we present a proof more in the spirit of these notes, due to Croke and Sharafutdinov in [**CS98**]. This proof generalizes to higher dimensions, whereas the earlier proof does not.

PROOF. Firstly we quote the result of Guillemin and Duistermaat [**DG75**] that the assumption (12.2.1) implies that we also have equality of the length spectrums, that is,

$$\mathrm{LSpec}(g_s) = \mathrm{LSpec}(g_0) \text{ for all } s \in (-\varepsilon, \varepsilon).$$

Let $\Pi(M)$ denote the set of all non-trivial free homotopy classes of $\pi_1(M)$. The function $s \mapsto L_{g_s}(\pi)$ for each $\pi \in \Pi(M)$ can be shown to be smooth. Since $\mathrm{LSpec}(g_s) = \mathrm{Im}\, L_{g_s}$ is countable, we can actually conclude the stronger result that the marked length spectrums coincide, that is,

$$L_{g_s} = L_s \text{ for all } s \in (-\varepsilon, \varepsilon). \tag{12.2.2}$$

Now fix $\pi \in \Pi(M)$, and let γ_s denote the closed geodesic in π for g_s, with $|\dot\gamma_s|_s = 1$. Then if $\ell = \ell_0(\gamma_0)$ then we have each γ_s defined on $[0, \ell]$, by (12.2.2). In particular, the *energy* of γ_s with respect to g_s is constant:

$$E_s(\gamma_s) := \int_0^\ell |\dot\gamma_s(t)|_s^2 \, dt = \ell \quad \text{for all} \quad s \in (-\epsilon, \epsilon).$$

Thus the argument of Proposition 2.6 shows that if $s_0 \in (-\varepsilon, \varepsilon)$ and

$$\beta_{s_0} := \frac{\partial g_s}{\partial s}\Big|_{s=s_0},$$

then β_{s_0} is a symmetric 2-tensor, and

$$I[\beta_{s_0}] = 0,$$

where I is the X-ray transform with respect to g_{s_0}.

Thus by Theorem 12.2 we conclude that there exists a family of smooth vector field Z_s such that

$$\beta_s(v, w) = \langle \nabla_v Z_s, w\rangle_s + \langle v, \nabla_w Z_s\rangle_s .$$

Then if we take ψ_s to be the flow of Z_s then $g_s = \psi_s^* g_0$. It remains to check that the map $s \mapsto \psi_s$ is smooth.

EXERCISE 12.8. Show that $s \mapsto \psi_s$ is smooth *(hint: use Theorem* 6.18*)*.

□

REMARK 12.9. In view of Exercise 12.4, Theorem 12.7 holds under the assumption that the geodesic flow is Anosov and $K \leq 0$.

We now outline the original approach to this result.

REMARK 12.10. *Guillemin and Kazhdan's approach to Theorem 12.7*: We will now briefly outline the original proof of Theorem 12.7. The reader is referred to [**GK80**] for the full details. We will see the same idea crop up again in the non-commutative setting of the next chapter (see Definition 13.15).

Guillemin and Kazhdan first defined

$$\eta^+ := \frac{X - iH}{2}, \qquad \eta^- := \frac{X + iH}{2},$$

so that

$$X = \eta^+ + \eta^-,$$

and then computed the commutation relations

$$[-iV, \eta^+] = \eta^+,$$

$$[-iV, \eta^-] = -\eta^-,$$

$$[\eta^+, \eta^-] = \frac{iKV}{2}.$$

Then they note that as an operator on $L^2(SM, \mathbb{C})$, V gives a decomposition

$$L^2(SM, \mathbb{C}) = \bigoplus_{n \in \mathbb{Z}} H_n,$$

where if $f \in H_n$ then $V(f) = inf$, and that $\eta^+ : H_n \to H_{n+1}$ and $\eta^- : H_n \to H_{n-1}$ are *first order elliptic operators*.

For example, if θ is a 1-form then $\theta = \theta_{-1} + \theta_1 \in H_{-1} \oplus H_1$, given by

$$\theta_1 = \frac{\theta - iV(\theta)}{2}, \qquad \theta_{-1} = \frac{\theta + iV(\theta)}{2},$$

so $\overline{\theta_1} = \theta_{-1}$. Similarly if β is a symmetric 2-tensor then $\beta = \beta_{-2} + \beta_0 + \beta_2$ with $\overline{\beta_2} = \beta_{-2}$ and $\overline{\beta_0} = \beta_0$. We want to solve

$$X(u) = \beta,$$

and writing $u = \sum_{n \in \mathbb{Z}} u_n$ with $u_n \in H_n$ we can write this as

$$(\eta^+ + \eta^-) \left(\sum_{n \in \mathbb{Z}} u_n \right) = \beta_{-2} + \beta_0 + \beta_2.$$

This decouples into the recurrence relation

$$\eta^-(u_{i+1}) + \eta^+(u_{i-1}) = \beta_i.$$

Using $K < 0$ together with $[\eta^+, \eta^-] = \frac{iKV}{2}$ one can show that $u_i = 0$ for $|i| \geq 2$ and from here it follows easily that, up to a constant, u is a 1-form as required. Hence β_s is potential, and ellipticity is then used to show that $s \mapsto \beta_s$ is smooth, and then the proof is completed as before.

We conclude this chapter by showing how the operators $\eta^{\pm}$ give us another derivation of the integrated Pestov identity from Corollary 9.9. This calculation shows how both approaches are related and why both are successful in dealing with this circle of questions.

Suppose $f \in C^\infty(SM, \mathbb{C})$, and write $f = u + iv$. Then we claim:

$$\left|\eta^+(f)\right|^2 = |\eta^-(f)|^2 + \frac{1}{2} \langle KiV(f), f \rangle; \tag{12.2.3}$$

Indeed, since $L_X \Theta = L_H \Theta = 0$ (Lemma 9.2), we also have $L_{\eta^\pm} \Theta = 0$, and thus

$$\int_{SM} \eta^+(\eta^-(f) \cdot \bar{f}) - \eta^-(\eta^+(f) \cdot \bar{f}) d\mu = 0.$$

Hence

$$\int_{SM} \eta^+\eta^-(f) \cdot \bar{f} - \eta^-\eta^+(f) \cdot \bar{f} d\mu = \int_{SM} \eta^+(f)\eta^-(\bar{f}) - \eta^-(f)\eta^+(\bar{f}) d\mu.$$

Using the fact that

$$\eta^+(\bar{f}) = \eta^+(u - iv) = \overline{\eta^-(u + iv)} = \overline{\eta^-(f)},$$

this becomes

$$\int_{SM} [\eta^+, \eta^-](f) \cdot \bar{f} d\mu = \int_{SM} \eta^+(f)\overline{\eta^+(f)} - \eta^-(f)\overline{\eta^-(f)} d\mu,$$

that is,

$$\langle [\eta^+, \eta^-](f), f \rangle = \left|\eta^+(f)\right|^2 - |\eta^-(f)|^2,$$

which is (12.2.3) since $[\eta^+, \eta^-](f) = \frac{iKV(f)}{2}$.

Now

$$\begin{aligned}\left|\eta^+(f)\right|^2 &= \frac{1}{4}\left|X(u)+H(v)+i\left\{X(v)-H(u)\right\}\right|^2 \\ &= \frac{1}{4}\int_{SM}\big\{[X(u)]^2+[H(v)]^2+2X(u)\cdot H(v) \\ &\quad +[X(v)]^2+[H(u)]^2-2X(v)\cdot H(u)\big\}\,d\mu,\end{aligned}$$

and similarly

$$\begin{aligned}|\eta^-(f)|^2 &= \frac{1}{4}\int_{SM}\big\{[X(u)]^2+[H(v)]^2-2X(u)\cdot H(v) \\ &\quad +[X(v)]^2+[H(u)]^2+2X(v)\cdot H(u)\big\}\,d\mu.\end{aligned}$$

Thus (12.2.3) gives us:

$$\begin{aligned}\int_{SM}\{X(u)\cdot H(v)-X(v)\cdot H(u)\}\,d\mu &= \left(\left|\eta^+(f)\right|^2-|\eta^-(f)|^2\right) \\ &= \frac{1}{2}\langle KiV(f),f\rangle \\ &= \frac{1}{2}\int_{SM}K\{(iV(u)-V(v))\cdot(u-iv)\}\,d\mu \\ &= \frac{1}{2}\int_{SM}K\{(vV(u)-uV(v)) \\ &\quad +i(uV(u)+vV(v))\}\,d\mu.\end{aligned}$$

Next, using the fact that

$$\begin{aligned}2vV(u)-V(uv) &= vV(u)-uV(v), \\ \frac{1}{2}\left[V(u^2)+V(v^2)\right] &= uV(u)+vV(v),\end{aligned}$$

the imaginary part disappears and we can rewrite this as

$$\int_{SM}\{X(u)\cdot H(v)-X(v)\cdot H(u)\}\,d\mu=\int_{SM}KvV(u)d\mu.$$

Now take $v=V(u)$ to obtain

$$\int_{SM}\{X(u)\cdot HV(u)-XV(u)\cdot H(u)\}\,d\mu=\int_{SM}K\,[V(u)]^2\,d\mu,$$

and then by the commutation relations

$$[V,X]=H,\qquad [H,V]=X,$$

this becomes

$$\int_{SM}[X(u)]^2+X(u)\cdot VH(u)-VX(u)\cdot H(u)+[H(u)]^2d\mu=\int_{SM}K[V(u)]^2d\mu.$$

Finally using

$$V\,(X(u)\cdot H(u))=VX(u)\cdot H(u)+X(u)\cdot VH(u),$$

we can rewrite this last line as

$$\int_{SM}[X(u)]^2-2VX(u)\cdot H(u)+[H(u)]^2d\mu=\int_{SM}K[V(u)]^2d\mu,$$

that is,

$$2\int_{SM}H(u)\cdot VX(u)d\mu=\int_{SM}[X(u)]^2\,d\mu+\int_{SM}[H(u)]^2\,d\mu-\int_{SM}K\,[V(u)]^2\,d\mu,$$

which is precisely the integrated Pestov identity from Corollary 9.9 (with $\lambda=0$).

CHAPTER 13

Transparent connections

In this last chapter we will discuss a problem in a similar vein to the problems discussed earlier related to the Kernel of the X-ray transform on 1-forms. This chapter is entirely based on the second author's recent papers [**Pat09b, Pat09a**], to which the reader is referred to for more information. In this chapter we will need rather more background material on connections and gauge theory than has been required up to now; thus for the benefit of the reader we begin with a quick summary of some of this material.

13.1. Review on connections and curvature

We begin with a quick review on some elementary material on connections that we will need throughout this chapter. A good reference for this material, written from a similar point of view is Chapter 3 of [**Jos08**], or, at a rather more advanced level, Chapter 2 of [**DK90**].

Let M denote a smooth manifold, and let $p : E \to M$ denote a complex vector bundle of rank n over M, with Hermitian inner product $h = \langle \cdot, \cdot \rangle$. Let $\Omega^r(E)$ denote the sheaf

$$U \mapsto \Omega^r(U, E) := \Gamma(U, \Lambda^r(T^*M) \otimes E), \quad U \subseteq M.$$

A *connection* ∇ on E is an $\mathbb{R}$-linear sheaf morphism $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying

$$\nabla(fs) = df \wedge s + f\nabla s \tag{13.1.1}$$

for local sections s of E and local smooth functions f. Let $\mathscr{X}(M)$ denote the sheaf of vector fields on M. Given a local vector field $X \in \mathscr{X}(U, M)$ we let ∇_X denote the sheaf morphism $\Omega^0(E)|_U \to \Omega^0(E)|_U$ given by

$$\nabla_X s := \nabla s(X).$$

We say that ∇ is *unitary* if it is compatible with the Hermitian product in the sense that

$$X \langle s, s' \rangle = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle$$

for all local vector fields X and local sections s, s' of E.

Let $\Omega^r(\operatorname{End} E)$ denote the sheaf of sections of the bundle $\operatorname{End} E$, and let $\Omega^r(\operatorname{ad} E)$ denote the subsheaf of $\Omega^r(\operatorname{End} E)$ such that $\Omega^r(U, \operatorname{ad} E)$ consists precisely of the elements $A \in \Omega^r(U, E)$ such that for all $x \in U$ and all $X_1, \dots, X_r \in T_x M$, the element

$$A(X_1, \dots, X_r) \in GL(E_x)$$

is skew-hermitian.

Suppose E is given by the cocycle $\{U_\alpha, f_{\alpha\beta}\}$, that is, $\{U_\alpha\}$ is an open cover of M such that $E|U_\alpha$ is trivial for all α, and $f_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$ are the transition functions for E. Then a connection ∇ on E determines elements $A_\alpha \in \Omega^1(U_\alpha, \operatorname{End} E)$ such that for $s \in \Omega^0(U_\alpha, E)$ we have

$$\nabla s = ds + A_\alpha s.$$

The elements $\{A_\alpha\}$ are related by:

$$A_\beta = f_{\alpha\beta}^{-1} df_{\alpha\beta} + f_{\alpha\beta}^{-1} A_\alpha f_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta.$$

If ∇ is unitary then the A_α are skew-symmetric and thus determine elements of $\Omega^1(U_\alpha, \operatorname{ad} E)$.

If ∇^1 and ∇^2 are two connections on E generating elements $\{A_\alpha^1\}$ and $\{A_\alpha^2\}$ respectively then if $A_\alpha := A_\alpha^2 - A_\alpha^1$ we have

$$A_\beta = f_{\alpha\beta}^{-1} A_\alpha f_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta,$$

and hence the $\{A_\alpha\}$ glue together to define a global element $A \in \Omega^1(M, \operatorname{End} E)$. If both ∇^1 and ∇^2 are unitary, then $A \in \Omega^1(M, \operatorname{ad} E)$, and hence we have shown:

LEMMA 13.1. *Given two unitary connections ∇^1 and ∇^2 on E there exists $A \in \Omega^1(M, \operatorname{ad} E)$ such that*

$$\nabla^2 = \nabla^1 + A.$$

EXERCISE 13.2. Suppose $E = M \times \mathbb{C}^n$ is the trivial bundle over M. Prove that if ∇ is any unitary connection on E then we can write $\nabla = d + A$ where $A \in \Omega^1(M, \operatorname{ad} E)$.

Slightly more generally, suppose we are given a Lie subgroup $G \leq GL(n, \mathbb{C})$. Given a vector bundle $p : E \to M$, we say that the *structure group of E can be reduced to G* if we can find a cocycle $\{U_\alpha, f_{\alpha\beta}\}$ for E with $f_{\alpha\beta}(x) \in G$ for all $x \in U_\alpha \cap U_\beta$. We then call the cocycle $\{U_\alpha, f_{\alpha\beta}\}$ a *G-cocycle*.

Let $p : E \to M$ be a complex rank n vector bundle, and suppose we can reduce its structure group to the Lie subgroup $G \leq GL(n, \mathbb{C})$. It therefore makes sense to talk about the *bundle of G-frames*. This is a principal G-bundle over M defined as follows. Let F_x denote the set of all bases $b = (e_1, \dots, e_n)$ of E_x such that as a matrix $b \in GL(n, \mathbb{C})$ we actually have $b \in G$. Thus $F_x \cong G$. Let $F := \bigsqcup_{x \in M} F_x$ and let $p_F : F \to M$ denote the projection sending $b \in F_x$ to $x \in M$. As in the case of the tangent bundle, we write an element of F in the form (x, b) to indicate that $b \in F_x$. We can define a right action of G on F by

$$F \times G \to F, \quad ((x, b), u) \mapsto (x, b \cdot u),$$

where '$\cdot$' denotes matrix multiplication.

Let $\mathfrak{g}$ denote the Lie algebra of G and form the set $\mathfrak{g}_E$ defined to be the quotient of $F \times \mathfrak{g}$ under the equivalence relation

$$((x, b), X) \sim ((x, b \cdot u), uXu^{-1}) \quad \text{for all } ((x, b), X) \in F \times \mathfrak{g} \text{ and } u \in G.$$

Then $\mathfrak{g}_E$ admits the structure of a vector bundle over M, and is a real subbundle of $\operatorname{End} E$. Let $\Omega^r(\mathfrak{g}_E)$ denote the sheaf of sections of this bundle. The previous bundle $\operatorname{ad} E$ is just the special case of $G = U(n)$, i.e., $\operatorname{ad} E = \mathfrak{u}(n)_E$.

A connection ∇ on E is called a *G-connection* if in a G-cocycle $\{U_\alpha, f_{\alpha\beta}\}$, ∇ is given by 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}_E)$. Exactly the same argument as in Lemma 13.1 shows that if we have two G-connections ∇^1 and ∇^2 then there exists $A \in \Omega^1(M, \mathfrak{g}_E)$ such that $\nabla^2 = \nabla^1 + A$. Thus unitary connections are precisely $U(n)$-connections. Later in this chapter we will focus exclusively on the case $G = SU(2)$.

Now suppose that $\psi : N \to M$ is a smooth map from another smooth manifold N. Then there is a bundle $\psi^* E$ over N defined as follows:

$$\begin{array}{ccc} \psi^* E & \xrightarrow{\bar{\psi}} & E \\ {\scriptstyle p_\psi}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow[\psi]{} & M \end{array}$$

Here

$$\psi^* E = \{(x, \xi) \in N \times E : \psi(x) = p(\xi)\},$$

and $p_\psi : (x, \xi) \mapsto x$ and $\bar{\psi} : (x, \xi) \mapsto \xi$ are the first and second projections. If E is given by cocycle $\{U_\alpha, f_{\alpha\beta}\}$ then $\psi^* E$ is given by cocycle $\{\psi^{-1}(U_\alpha), f_{\alpha\beta} \circ \psi\}$.

Suppose ∇ is a connection on E. Then we have a unique connection $\psi^* \nabla$ on $\psi^* E$ defined by

$$(\psi^* \nabla)_X (\psi^* s) = \nabla_{d\psi(X)} s, \quad X \in \mathscr{X}(N), \ s \in \Omega^0(E).$$

Alternatively, if ∇ determines elements $A_\alpha \in \Omega^1(U_\alpha, \operatorname{End} E)$ then $\psi^* \nabla$ determines the elements $\psi^* A_\alpha \in \Omega^1(\psi^{-1}(U_\alpha), \operatorname{End} \psi^* E)$, that is,

$$\psi^* \nabla|_{\psi^{-1}(U_\alpha)} = d + f^* \psi_\alpha$$

is the local description of $\psi^* \nabla$.

Given a connection ∇ on E, there exists a unique connection ∇^* on E^* determined as follows: for $\sigma \in \Omega^0(E^*)$ and $s \in \Omega^0(E)$,

$$\nabla^*_X \sigma(s) := \nabla_X(\sigma(s)) - \sigma(\nabla_X s).$$

This in turn defines a connection ∇^{End} on $E \otimes E^* = \operatorname{End} E$ by

$$\nabla^{\mathrm{End}}_X (s \otimes \sigma) = \nabla_X s \otimes \sigma + s \otimes \nabla^*_X \sigma.$$

Let $\{e_1, \ldots, e_n\}$ be a local frame for E over U_α, and let $\{\varepsilon^1, \ldots, \varepsilon^n\}$ be the corresponding dual frame of E^*. Then a local section $u \in \Omega^0(U_\alpha, E \otimes E^*)$ of $\operatorname{End} E$ can be written as

$$u = \tau^i_j e_i \otimes \varepsilon^j, \quad \tau^i_j \in C^\infty(U_\alpha, \mathbb{C}).$$

If A_α has entries $a^i_j \in \Omega^1(U_\alpha, M)$ then

$$\begin{aligned} \nabla^{\mathrm{End}} u &= d\tau^i_j e_i \otimes \varepsilon^j + \tau^i_j a^k_i e_k \otimes \varepsilon^j - \tau^i_j a^j_k e_i \otimes \varepsilon^k \\ &= du + [A_\alpha, u]. \end{aligned}$$

Thus if $\nabla|_{U_\alpha} = d + A_\alpha$ then $\nabla^{\mathrm{End}}|_{U_\alpha} = d + [A_\alpha, \cdot]$. Taking this one step further and considering the induced connection $(\psi^*\nabla)^{\mathrm{End}}$ on $\operatorname{End} \psi^* E$, we see that $(\psi^*\nabla)^{\mathrm{End}}|_{\psi^{-1}(U_\alpha)} = d + [\psi^* A_\alpha, \cdot]$.

Now we define the *curvature* of a connection ∇. Firstly, there exists a unique extension of ∇ to a map $d^\nabla : \Omega^r(E) \to \Omega^{r+1}(E)$ defined as follows: if ω is a local r-form on M and s is a local section of E then

$$d^\nabla(\omega \otimes s) := d\omega \otimes s + (-1)^r \omega \wedge \nabla s.$$

For $r = 0$, this is just the defining equation (13.1.1) for the connection, which ensures that d^∇ is well defined, that is, $d^\nabla(f\omega \otimes s) = d^\nabla(\omega \otimes fs)$ for any smooth function f. Next, we define the curvature F_∇ of ∇ to be the composition

$$F_\nabla = d^\nabla \circ d^\nabla : \Omega^0(E) \to \Omega^2(E).$$

One easily checks that F_∇ is $C^\infty(M, \mathbb{R})$-linear, that is,

$$F_\nabla(fs) = f F_\nabla s,$$

and hence F_∇ determines a global section $F_\nabla \in \Omega^2(M, \operatorname{End} E)$. If ∇ is unitary then it is easily seen that $F_\nabla \in \Omega^2(M, \operatorname{ad} E)$.

If $\nabla|_{U_\alpha} = d + A_\alpha$ then a trivial calculation shows that

$$F_\nabla|_{U_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha;$$

note that this shows that if E is a line bundle then $F_\nabla|_{U_\alpha} = dA_\alpha$, and hence in this case F_∇ is simply an ordinary 2-form, $F_\nabla \in \Omega^2(M)$.

The curvature is natural with respect to pullbacks, that is, if $\psi : N \to M$ is smooth and $\psi^*\nabla$ is the induced connection on $\psi^* E$ then

$$F_{\psi^*\nabla} = \psi^* F_\nabla \in \Omega^2(N, \operatorname{ad} \psi^* E).$$

Indeed

$$\begin{aligned} F_{\psi^*\nabla}|_{\psi^{-1}(U_\alpha)} &= d(\psi^* A_\alpha) + \psi^* A_\alpha \wedge \psi^* A_\alpha \\ &= \psi^*(dA_\alpha + A_\alpha \wedge A_\alpha) \\ &= \psi^*(F_\nabla|_{U_\alpha}). \end{aligned}$$

Similarly one checks

$$F_{\nabla^{\mathrm{End}}} = [F_\nabla, \cdot],$$

and thus we have

$$F_{(\psi^*\nabla)^{\mathrm{End}}} = \left[\psi^* F_\nabla, \cdot\right].$$

Finally we discuss gauge transformations. Let $\mathcal{G}_E$ denote the global sections of the bundle $\operatorname{Aut} E$, that is,

$$\begin{aligned} \mathcal{G}_E &:= \Omega^0(M, \operatorname{Aut} E) \\ &= \left\{\omega \in \Omega^0(M, \operatorname{End} E) : \omega_x : E_x \to E_x \text{ is an isomorphism for all } x \in M\right\}. \end{aligned}$$

We call $\mathcal{G}_E$ the *gauge group* of E, and elements of $\mathcal{G}_E$ are called *gauge transformations*. The group structure on $\mathcal{G}_E$ is given by fibrewise matrix multiplication. A gauge transformation ω operates on the space of unitary connections by

$$\nabla \mapsto \omega^* \nabla,$$

where for $s \in \Omega^0(E)$,

$$(\omega^*\nabla)(s) := \omega^{-1}(\nabla \omega(s)).$$

Alternatively if ∇ is given by $d + A_\alpha$ on U_α then $\omega^*\nabla$ is given by

$$d + (\omega_\alpha^{-1} d\omega_\alpha + \omega_\alpha^{-1} A_\alpha \omega_\alpha) \text{ on } U_\alpha,$$

where ω is given by ω_α on U_α, so $\omega_\beta = f_{\alpha\beta}^{-1}\omega_\alpha f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ if $\{f_{\alpha\beta}\}$ are the transition functions associated to the cover $\{U_\alpha\}$.

We say that two unitary connections ∇^1 and ∇^2 are *gauge equivalent* if there exists $\omega \in \mathcal{G}_E$ such that $\nabla^2 = \omega^*\nabla^1$. Gauge equivalence is clearly an equivalence relation. We let $\mathcal{C} = \mathcal{C}(E)$ denote the space of unitary connections on E modulo gauge equivalence.

13.2. Bundle-valued cocycles

We now need to generalize Definition 6.5 and *bundle-valued cocycles*.

DEFINITION 13.3. Let $\phi_t : N \to N$ be a flow, where N is a closed manifold. Let $p : E \to N$ be a complex Hermitian vector bundle over N. A *E-valued cocycle C over ϕ_t* is a map C such that for each $(x, t) \in N \times \mathbb{R}$, $C(x, t) : E_x \to E_{\phi_t x}$ is a unitary map satisfying

$$C(x, t + s) = C(\phi_t x, s) \circ C(x, t) : E_x \to E_{\phi_{t+s} x}.$$

If E is trivial, with $\Phi : E \to N \times \mathbb{C}^n$ a unitary trivialization, then we can recover a cocycle D in the sense of Definition 6.5 as follows: given $(x, t) \in N \times \mathbb{R}$, and $a \in \mathbb{C}^n$, we can write

$$\Phi \circ C(x, t) \circ \Phi^{-1}(x, a) = (\phi_t x, b) \in N \times \mathbb{C}^n$$

for some $b \in \mathbb{C}^n$; define $D : N \times \mathbb{R} \to U(n)$ by

$$D(x, t)a = b,$$

that is,

$$\Phi \circ C(x, t) \circ \Phi^{-1}(x, a) = (\phi_t x, D(x, t)a).$$

It is then elementary to check that D is a $U(n)$-valued cocycle over ϕ_t in the sense of Definition 6.5.

Given such an E-valued cocycle C over ϕ_t, we can construct an E^*-valued cocycle C^* over ϕ_t as follow, where E^* is the dual vector bundle. Let $\hat{h} : E \xrightarrow{\sim} E^*$ denote the conjugate isomorphism induced by the Hermitian metric, that is,

$$\hat{h}(\xi)(\eta) := \langle \eta, \xi \rangle, \quad \xi, \eta \in E.$$

Then define C^* by

$$C^*(x, t) := \hat{h} \circ C(x, t) \circ \hat{h}^{-1} : E_x^* \to E_{\phi_t x}^*.$$

Similarly given two bundles E and E' over N, with corresponding cocycles C and C' over ϕ_t, we can define an $E \oplus E'$-valued cocycle $C \oplus C'$ over ϕ_t in the obvious way.

Finally, we have the analogue of the periodic orbit obstruction condition from Definition 6.7. Namely, an E-valued cocycle C over ϕ_t satisfies the *periodic orbit obstruction condition* if whenever $\phi_T x = x$ we have

$$C(x, T) = \mathrm{Id} : E_x \to E_x.$$

It is easy to see that if E is trivial, then C satisfies the periodic orbit obstruction condition if and only if the associated $U(n)$-valued cocycle D constructed above satisfies the periodic orbit obstruction condition in the sense of Definition 6.7.

Here is the first result we need on E-valued cocycles.

PROPOSITION 13.4. *Let $\phi_t : N \to N$ be a transitive Anosov flow over a closed manifold N. Let $p : E \to N$ be a rank n Hermitian vector bundle such that $E \oplus E^*$ is a trivial vector bundle. Suppose there exists a E-valued cocycle C over ϕ_t satisfying the periodic orbit obstruction condition. Then E is a trivial vector bundle.*

PROOF. Let $\Phi : E \oplus E^* \to N \times \mathbb{C}^{2n}$ denote a unitary trivialization, and let $D : N \times \mathbb{R} \to U(2n)$ denote the cocycle over ϕ_t defined by

$$\Phi \circ C \oplus C^*(x, t) \circ \Phi^{-1}(x, a) = (\phi_t x, D(x, t)a) \in N \times \mathbb{C}^{2n}.$$

Then D satisfies the periodic orbit obstruction condition and hence by the Livsic Cocycle Regularity Theorem 6.12, we deduce the existence of a smooth function $u : N \to U(2n)$ such that

$$D(x,t) = u(\phi_t x)u(x)^{-1} \quad \text{for all } (x,t) \in N \times \mathbb{R}.$$

Now choose a point $x_0 \in N$ such that the orbit $\{\phi_t x_0 : t \in \mathbb{R}\}$ is dense in N (using transitivity of ϕ_t). Let $U := u(x_0) \in U(2n)$. Select a unitary basis $\{v_1, \ldots, v_n\}$ of E_{x_0}, and define points $a_i \in \mathbb{C}^{2n}$ by

$$\Phi(x_0, v_i) = (x_0, a_i) \in N \times \mathbb{C}^{2n}.$$

Now define sections $e_i : N \to E \oplus E^*$ by

$$e_i(x) = \Phi^{-1}(x, u(x)U^{-1}(a_i)).$$

It is clear that $\{e_1(x), \ldots, e_n(x)\}$ is a linearly independent set in $(E \oplus E^*)_x$ for all $x \in N$. In fact we claim that $e_i(x) \in E_x$ for all $i = 1, \ldots, n$ and $x \in N$. This of course implies the triviality of E.

To see this observe that

$$\begin{aligned} e_i(\phi_t x_0) &= \Phi^{-1}(\phi_t x_0, u(\phi_t x_0)U^{-1}(a_i)) \\ &= \Phi^{-1}(\phi_t x_0, D(x_0,t)(u(x_0)U^{-1}(a_i))) \\ &= \Phi^{-1}(\phi_t x_0, D(x_0,t)a_i) \\ &= C \oplus C^*(x_0,t)((v_i,0)) \in E_{\phi_t x_0}. \end{aligned}$$

It follows that $e_i(x) \in E_x$ for a dense set of points in N. By continuity of e_i, $e_i(x) \in E_x$ for all $x \in N$.

□

13.3. Transparent connections

As before let (M,g) be a closed Riemannian manifold and $p : E \to M$ a Hermitian complex vector bundle of rank n over M. Let ∇ be a unitary connection on E. Before giving the key definition of this chapter, let us quickly recall the concept of parallel transport. Let $\gamma : [0,T] \to M$ be a smooth curve in M. Let $\frac{D}{dt}$ denote covariant derivation along $\gamma(t)$. Suppose $\xi \in E_{\gamma(0)}$. Recall that standard ODE theory ensures that there exists a unique section s_ξ of E along γ satisfying $\frac{D}{dt}s_\xi \equiv 0$ and $s_\xi(0) = \xi \in E_{\gamma(0)}$. This defines a map $P_\gamma(t) : E_{\gamma(0)} \to E_{\gamma(t)}$ by

$$P_\gamma(t)(\xi) = s_\xi(t).$$

It is easy to see that $P_\gamma(t)$ is a linear unitary isomorphism. We call the family P_γ of isomorphisms the *parallel transport along* γ.

Here then is the key definition.

DEFINITION 13.5. Let (M,g) be a closed Riemannian manifold and $p : E \to M$ a Hermitian complex vector bundle of rank n over M. Let ∇ be a unitary connection on E. We say that ∇ is *transparent* if for every closed geodesic $\gamma : [0,T] \to M$ we have

$$P_\gamma(T) = \mathrm{Id} : E_{\gamma(0)} \to E_{\gamma(0)}.$$

EXAMPLE 13.6. Here are some examples of transparent connections.

(1) Here is the simplest possible example. Let E be the trivial vector bundle $M \times \mathbb{C}^n$, and let ∇ denote the trivial connection. Then ∇ is certainly transparent.
(2) Suppose M has dimension 2. Then M admits the structure of a closed Riemann surface. Let K be the canonical line bundle of M, which in this case is simply the holomorphic cotangent bundle, $K = T^*M^{1,0}$. Let ∇^{lc} denote the unitary connection induced on E by the Levi-Civita connection on M. Then ∇^{lc} is evidently transparent.
(3) Generalizing this, let K^s denote the sth tensor product of K for $s \in \mathbb{Z}$, where K^0 is understood to be the trivial bundle. Let ∇^{lc}_s denote the induced connection on K^s. Then ∇^{lc}_s is a transparent connection on K^s.
(4) Going even further, given an n-tuple of integers $S = (s_1, \ldots, s_n)$, let

$$E_S := K^{s_1} \oplus \cdots \oplus K^{s_n}$$

and let

$$\nabla^{lc}_S := \nabla^{lc}_{s_1} \oplus \cdots \oplus \nabla^{lc}_{s_n}$$

denote the induced connection on E_S. Then ∇^{lc}_S is a transparent connection on E_S.

(5) Recall that complex vector bundles E over surfaces are classified (topologically) by their first Chern class $c_1(E)$ (see for instance [**GH78**, p140]) . The first Chern class of the canonical line bundle K is $2\mathrm{g}-2$, where g is the genus of the surface M, and the first Chern class of K^s is $(2\mathrm{g}-2)s$, and the first Chern class of E_S is $(2\mathrm{g}-2)(s_1+\cdots+s_n)$. Suppose now E is any Hermitian bundle over M such that $2\mathrm{g}-2$ divides $c_1(E)$. Then we have $c_1(E)=c_1(E_S)$ for some S, and hence there exists a unitary isomorphism $\Psi : E \to E_S$. Then $\Psi^*\nabla_S^{lc}$ is a transparent connection on E.

Thus for surfaces, we have shown that if a Hermitian bundle E has first Chern class dividing $2\mathrm{g}-2$ (with g the genus of the surface) then E admits a transparent connection. In fact, this is a necessary condition:

PROPOSITION 13.7. *Let (M,g) be a closed Riemannian surface of genus* g *whose geodesic flow is Anosov, and let $p : E \to M$ be a Hermitian vector bundle over M admitting a transparent connection ∇. Then $2\mathrm{g}-2$ divides $c_1(E)$.*

PROOF. This is a simple application of the Gysin sequence of the unit circle bundle $\pi : SM \to M$ from Proposition 8.8 together with Proposition 13.4. Indeed, since $E\oplus E^*$ is trivial as $c_1(E\oplus E^*) = c_1(E)-c_1(E) = 0$, Proposition 13.4 implies that π^*E is trivial, and since $c_1(\pi^*E) = \pi^*c_1(E)$, we conclude $\pi^*c_1(E)=0$. Then the Gysin sequence

$$0 \to H^1(M,\mathbb{Z}) \xrightarrow{\pi^*} H^1(SM,\mathbb{Z}) \to H^0(M,\mathbb{Z}) \xrightarrow{\times 2-2\mathrm{g}} H^2(M,\mathbb{Z}) \xrightarrow{\pi^*} H^2(SM,\mathbb{Z}) \to \dots.$$

shows that $\pi^*c_1(E)=0$ if and only if $c_1(E)$ is in the image of the map $H^0(M,\mathbb{Z})\to H^2(M,\mathbb{Z})$ given by cup product with the Euler class of SM. Equivalently, this happens if and only if $2-2\mathrm{g}$ divides $c_1(E)$. □

The situation is simpler in three dimensions, as the following exercise shows.

EXERCISE 13.8. Suppose (M,g) is a closed 3-manifold whose geodesic flow is Anosov, and $p : E \to M$ be a Hermitian vector bundle over M admitting a transparent connection ∇. Prove that E is trivial.

Let us suppose now that $E_0 = M\times\mathbb{C}^n$ is the trivial vector bundle over the closed Riemannian manifold (M,g). Let $\phi_t : SM \to SM$ denote the geodesic flow, and let ∇ denote a unitary connection on E_0. Then ∇ defines a cocycle over ϕ_t in a natural way. Namely, in this case using Exercise 13.2 we can write $\nabla = d + A$ where $A : TM \to \mathfrak{u}(n)$ is a smooth function that is linear in $v\in T_xM$ for all $x\in M$. Now define $C : SM\times\mathbb{R}\to U(n)$ by the ODE

$$\frac{d}{dt}C(x,v,t) := -A(\phi_t(x,v))C(x,v,t), \quad C(x,v,0)=\mathrm{Id}.$$

Observe that ∇ is a *transparent connection* if and only if $C(x,v,T)=\mathrm{Id}$ whenever $\phi_T(x,v)=(x,v)$.

We will say that the cocycle C is *cohomologically trivial* if there exists a smooth function $u : SM \to U(n)$ such that

$$C(x,v,t) = u(\phi_t(x,v))u(x,v)^{-1}. \tag{13.3.1}$$

Clearly if C is cohomologically trivial then the connection ∇ is transparent. Conversely, when ϕ_t is Anosov, we have the following theorem.

THEOREM 13.9. *If ϕ_t is Anosov then ∇ is transparent if and only if C is cohomologically trivial.*

PROOF. This is an application of the "full" Livsic Periodic Data Theorems from Chapter 6. Indeed, if ϕ_t is Anosov and ∇ is transparent then Theorem 6.8 gives the existence of a Hölder continuous function $v : SM \to U(n)$ satisfying (13.3.1), and then Theorem 6.12 allows us to upgrade v to a *smooth* function u satisfying (13.3.1). The proof is complete. □

13.4. The space $\mathcal{T}$ of transparent connections

Let us introduce the following notation. Let $p : E \to M$ be a Hermitian vector bundle, and as before let $\mathcal{C} = \mathcal{C}(E)$ denote the space of connections modulo gauge equivalence. Consider the space $\mathcal{T}\subseteq\mathcal{C}$ of transparent connections modulo gauge equivalence. The main goal of this chapter is to obtain some kind of measure as to the size of $\mathcal{T}$, at least in the case where M is a closed surface with Anosov geodesic flow.

For the case of line bundles, $\mathcal{T}$ is rather small. Indeed, we have the following result:

THEOREM 13.10. *Let (M, g) be a closed Riemannian surface with Anosov geodesic flow, and let $p : E \to M$ be a Hermitian line bundle over M. Then any two transparent connections on E are gauge equivalent, that is, $\mathcal{T}(E)$ consists of at most one point.*

PROOF. Let ∇^1 and ∇^2 be two transparent connections on E. We may write $\nabla^2 = \nabla^1 + A$, where $A \in \Omega^1(M, \mathrm{ad}\, E)$. Since E is a line bundle, we may write $A = i\theta$, where θ is a real valued 1-form on M.

EXERCISE 13.11. Prove that for every closed geodesic γ on M,

$$\int_\gamma \theta \in 2\pi\mathbb{Z}.$$

Now let us define an S^1-valued cocycle C over ϕ_t by

$$C(x, v, t) := e^{i \int_0^t \theta(\dot\gamma_{(x,v)}(s))ds},$$

where $\gamma_{(x,v)}$ is the unique geodesic adapted to (x, v). Then clearly $\phi_T(x, v) = (x, v)$ implies $C(x, v, T) = 1$, and hence by Theorem 6.12 there exists a smooth function $u : SM \to S^1$ such that $C(x, v, t) = u(\phi_t(x, v))u(x, v)^{-1}$. Differentiating this relation and setting $t = 0$, we obtain

$$i\theta_x(v)u(x, v) = d_{(x,v)}u(X(x, v)), \tag{13.4.1}$$

where X is the infinitesimal generator of ϕ_t. Now consider the closed 1-form $\varphi := \frac{du}{iu} \in \Omega^1(SM)$. Since M is necessarily not the 2-torus (see Corollary 9.5), $\pi^* : H^1(M, \mathbb{R}) \to H^1(SM, \mathbb{R})$ is an isomorphism (see Corollary 8.10), and hence there exists a closed 1-form ω on M and a smooth function h on M such that

$$\varphi = \pi^*\omega + dh. \tag{13.4.2}$$

Applying X to both sides of (13.4.2) and using (13.4.1) we get

$$\theta_x(v) = \omega_x(v) + d_{(x,v)}h(X(x, v)).$$

This holds for all $(x, v) \in SM$, and hence

$$I[\theta - \omega] = 0,$$

where I is the X-ray transform. Then by Theorem 7.15, this implies that $\theta - \omega$ is exact, and so in particular, θ is closed. Then since $\int_\gamma \theta \in 2\pi\mathbb{Z}$ for all closed geodesics γ, we must have $\left[\frac{\theta}{2\pi}\right] \in H^1(M, \mathbb{Z})$, and so we may write

$$\theta = \frac{dg}{ig}$$

for some smooth function $g : M \to S^1$.

But this is precisely what we needed for ∇^1 and ∇^2 to be gauge equivalent; namely $\nabla^2 = \nabla^1 + A$ implies

$$\nabla^2 = \nabla^1 + g^{-1}dg. \tag{13.4.3}$$

Explicitly, if $\nabla^i|_{U_\alpha} = d + A^i_\alpha$ for $A^i_\alpha \in \Omega^1(U_\alpha, \mathrm{ad}\, E)$, and $g|_{U_\alpha} = g_\alpha$, where $g_\beta = \psi_{\beta\alpha} g_\alpha$, (13.4.3) implies that

$$A^2_\alpha = A^1_\alpha + g_\alpha^{-1} dg_\alpha.$$

But then

$$\begin{aligned} \nabla^2|_{U_\alpha} &= d + A^2_\alpha \\ &= d + (g_\alpha^{-1} dg_\alpha + A_\alpha) \\ &= d + (g_\alpha^{-1} dg_\alpha + g_\alpha^{-1} A_\alpha g_\alpha), \end{aligned}$$

where the last line follows as

$$g_\alpha^{-1} A^1_\alpha g_\alpha = A^1_\alpha,$$

since $GL(1, \mathbb{C})$ is commutative. Thus $\nabla^2 = g^*\nabla^1$ as required. □

In order to cope with bundles of rank greater than one, considerable more work is required. For the remainder of this section, we take (M, g) to be a closed Riemannian surface, $p : E \to M$ is a Hermitian complex vector bundle of rank n, and ∇ a unitary connection (which will often be taken to be transparent). If $\pi : SM \to M$ denotes the footpoint map, we have an induced connection $\pi^*\nabla$ on the pullback bundle π^*E, and then another induced connection $D := (\pi^*\nabla)^{\mathrm{End}}$ on $\operatorname{End}\pi^*E$.

The bundle $\operatorname{End}\pi^*E$ inherits an L^2 inner product from the Hermitian metric from E as follows. To begin with, pull $\langle\cdot,\cdot\rangle$ back via the second projection $\bar{\pi} : \pi^*E \to E$ to obtain a Hermitian metric (also denoted by) $\langle\cdot,\cdot\rangle$ on π^*E. We can use this to define the adjoint u^* of an element $u \in \operatorname{End}\pi^*E$; namely by

$$\langle u(\xi), \eta\rangle = \langle \xi, u^*(\eta)\rangle,$$

and this defines a Hermitian metric $(u, w) := \operatorname{tr}(uw^*)$ on $\operatorname{End}\pi^*E$. This metric is compatible with D:

$$Y(u, w) = (D_Y u, w) + (u, D_Y w) \tag{13.4.4}$$

for a vector field Y on SM and sections $u, w \in \Omega^0(SM, \operatorname{End}\pi^*E)$.

Finally we define the desired L^2-inner product $\langle\cdot,\cdot\rangle_{L^2}$ on sections u, w of the bundle $\operatorname{End}\pi^*E$ by

$$\langle u, w\rangle_{L^2} := \int_{SM} \operatorname{tr}(uw^*)d\mu,$$

where μ denotes the Liouville measure.

We have

$$\int_{SM} X(\operatorname{tr}(uw^*))d\mu = \int_{SM} H(\operatorname{tr}(uw^*))d\mu = \int_{SM} V(\operatorname{tr}(uw^*))d\mu = 0$$

by Lemma 9.2 and together with (13.4.4) we have the following lemma.

LEMMA 13.12. *Recall the framing $\{X, H, V\}$ of SM. The operators $-iD_X, -iD_H$ and $-iD_V$: $\Omega^0(SM, \operatorname{End}\pi^*E) \to \Omega^0(SM, \operatorname{End}\pi^*E)$ are self-adjoint with respect to $\langle\cdot,\cdot\rangle_{L^2}$.*

REMARK 13.13. Observe that the operator D_V is in fact independent of ∇, for if ∇' is another connection and we write $\nabla' = \nabla + A$, then $D' = D + [\pi^*A, \cdot]$ and $D'_V = D_V$ since $\pi^*A(V) = 0$.

The lemma above tells us almost immediately the following,

PROPOSITION 13.14. *There exists an orthogonal decomposition*

$$L^2(SM, \operatorname{End}\pi^*E) = \bigoplus_{m\in\mathbb{Z}} H_m,$$

such that $-iD_V = m\mathrm{Id}$ on H_m.

PROOF. Let us first consider the special case where both E and SM are trivial, and ∇ is the trivial connection. Thus $SM = M \times S^1$ and $E = M \times \mathbb{C}^n$, and hence $\pi^*E = M \times S^1 \times \mathbb{C}^n$. In this case $u \in \Omega^0(SM, \operatorname{End}\pi^*E)$ is simply a map $u : M \times S^1 \to GL(n, \mathbb{C})$ and $D_V u = V(u)$. Fix $x \in M$. Then if $u_x : S^1 \to GL(n, \mathbb{C})$ denotes the map $u_x(\theta) := u(x, \theta)$ then we have

$$V(u) = \frac{\partial}{\partial\theta}u_x(\theta).$$

The statement then reduces to the assertion that smooth functions on the circle admit a Fourier expansion; that is, $\{e^{ik\theta} : k \in \mathbb{Z}\}$ forms an orthonormal Hilbert basis of $L^2(S^1, \mathbb{C})$.

Now we proceed to the general case. Triangulate M in such a way that both E and SM are trivial over each face M_r of the triangulation. Since

$$L^2(SM, \operatorname{End}\pi^*E) = \bigoplus_r L^2(SM_r, \operatorname{End}\pi^*E),$$

the result then follows from the special case proved above. □

The next step is to introduce the following operators, following Guillemin and Kazhdan in [**GK80**].

DEFINITION 13.15. Define operators $\eta_+, \eta_- : \Omega^0(SM, \operatorname{End}\pi^*E) \to \Omega^0(SM, \operatorname{End}\pi^*E)$ by

$$\eta_+ := \frac{D_X - iD_H}{2},$$

$$\eta_- := \frac{D_X + iD_H}{2}.$$

The following easy lemma describes the commutation relations between $\eta_\pm$. In what follows, $\star$ denotes the Hodge star operator of the metric, and K denotes the Gaussian curvature of M.

LEMMA 13.16. *We have*

$$[-iD_V,\eta_+]=\eta_+,$$
$$[-iD_V,\eta_-]=-\eta_-,$$
$$[\eta_+,\eta_-]=\frac{i}{2}\left(KD_V+[\star F_\nabla,\cdot]\right).$$

PROOF. Firstly, since V is vertical, for any vector field U we have $F_D\,(V,U)=0$. Indeed,

$$\begin{aligned} F_D(V,U) &= \left[\pi^* F_\nabla(V,U),\cdot\right] \\ &= [F_\nabla(d\pi(V),d\pi(U)),\cdot] \\ &= 0, \end{aligned}$$

since $d\pi(V)=0$. Thus using

$$F_D\,(V,U)=D_VD_U-D_UD_V-D_{[V,U]},$$

and taking $U=X$, we have

$$[D_V,D_X]=D_{[V,X]}=D_H.$$

Taking $U=H$ we obtain $[D_V,D_H]=-D_X$. The first two equations are then immediate. To prove the third relation we start from the observation that

$$\begin{aligned} 2\,[\eta_+,\eta_-] &= i\,[D_X,D_H] \\ &= F_D(X,H)+KD_V, \end{aligned}$$

and then use that fact that for any $(x,v)\in SM$,

$$\begin{aligned} F_D(X,H)(x,v) &= \left[\pi^* F_\nabla(X,H)(x,v),\cdot\right] \\ &= [F_\nabla(d\pi(X)(x,v),d\pi(H)(x,v),\cdot] \\ &= [F_\nabla(v,\mathtt{i}v),\cdot]\,, \end{aligned}$$

which is equivalent to

$$F_D(X,H)=[\star F_\nabla,\cdot]\,.$$

From this the third relation easily follows. □

Set $\Omega_m:=H_m\cap\Omega^0(SM,\operatorname{End}\pi^*E)$. Then the first two relations above imply that η_+ maps Ω_m into Ω_{m+1}, and η_- maps Ω_m into Ω_{m-1}. Using Lemma 13.12 we see that $\eta_+^*=-\eta_-$ and $\eta_-^*=-\eta_+$.

Now suppose ∇^0 is another unitary connection on E. As before write $\nabla^0=\nabla+A$ for $A\in\Omega^1(M,\operatorname{ad}E)$. There are two ways we may think of A as an element of $\Omega^*(SM,\operatorname{ad}\pi^*E)$. Firstly, if $U\subseteq M$ is a trivializing neighborhood for E, and $\{e_1,\dots,e_n\}$ a local frame on E with coframe $\left\{\varepsilon^1,\dots,\varepsilon^n\right\}$ of E^*, then we can write $A|U$ as $a^i_je_i\otimes\varepsilon^j$ for some 1-forms $a^i_j\in\Omega^1_{\mathbb{C}}(U)$. Simply regarding the a^i_j as lying in $C^\infty(S_UM,\mathbb{C})$, we may think of $a^i_je_i\otimes\varepsilon^j$ as defining an element of $\Omega^0(SM,\operatorname{ad}\pi^*E)$, which we shall continue to write as A.

We can also form

$$\pi^*A=(\pi^*a^i_j)e_i\otimes\varepsilon^j\in\Omega^1(SM,\operatorname{ad}\pi^*E).$$

These are not the same thing! They are related by $(\pi^*A)(X)=A$: indeed

$$\begin{aligned} (\pi^*A)(X) &= (\pi^*a^i_j)(X)e_i\otimes\varepsilon^j \\ &= a^i_j(d\pi(X))e_i\otimes\varepsilon^j \\ &= A. \end{aligned}$$

Similarly $(\pi^*A)(H)=-\star A$.

Moreover we also have

$$D_VA(x,v)=A(x,\mathtt{i}v)=(-\star A)(x,v),$$

and thus $D_VA=(\pi^*A)(H)$. We also have $D_V^2A=-A$.

Let us decompose

$$A=A_{-1}+A_1$$

where

$$A_{-1} := \frac{A - iD_V A}{2} \in \Omega_1,$$

and

$$A_1 := \frac{A + iD_V A}{2} \in \Omega_{-1}.$$

Then finally set

$$\mu_+ := \eta_+ + A_1, \quad \mu_- := \eta_- + A_{-1}.$$

EXERCISE 13.17. Prove that $\mu_+ : \Omega_m \to \Omega_{m+1}$, $\mu_- : \Omega_m \to \Omega_{m-1}$ and $\mu_+^* = -\mu_-$, $\mu_-^* = -\mu_+$.

The following relations however are rather less immediate.

LEMMA 13.18. *It holds that*

(1) $\frac{i}{2} \star (\nabla A + A \wedge A) = \eta_+(A_{-1}) - \eta_-(A_1) + A_1 A_{-1} - A_{-1} A_1$,
(2) *Given* $u \in \Omega^0(SM, \operatorname{End}\pi^* E)$ *we have*

$$[\mu_+, \mu_-]\, u = \frac{i}{2}\left(KD_V u + (\star F_{\nabla^0})u - u(\star F_\nabla)\right),$$

(3) *If* $\|\cdot\|$ *stands for the* L^2 *norm on* $\Omega^0(SM, \operatorname{End}\pi^* E)$ *induced by* $\langle\cdot,\cdot\rangle_{L^2}$*, then for* $u \in \Omega^0(SM, \operatorname{End}\pi^* E)$ *we have*

$$\|\mu_+ u\|^2 = \|\mu_- u\|^2 + \frac{i}{2}\left\{\langle KD_V u, u\rangle_{L^2} + \langle(\star F_{\nabla^0})u, u\rangle_{L^2} - \langle u(\star F_\nabla), u\rangle_{L^2}\right\}.$$

PROOF. From the definitions we have

$$A_1 A_{-1} - A_{-1} A_1 = \frac{i}{2}\left(A \cdot D_V A - D_V A \cdot A\right)$$

and

$$\eta_+(A_{-1}) - \eta_-(A_1) = \frac{i}{2}\left(D_X D_V A - D_H A\right).$$

Next,

$$\star(A \wedge A) = A \cdot D_V A - D_V A \cdot A,$$

and then

$$\begin{aligned} \star(\nabla A) &= D(\pi^* A)(X, H) \\ &= D_X(\pi^* A(H)) - D_H(\pi^* A(X)) - (\pi^* A)([X, H]) \\ &= D_X D_V A - D_H A - 0. \end{aligned}$$

From this 1 follows.

To prove 2, first compute

$$\begin{aligned} [\mu_+, \mu_-]\, u &= [\eta_+, \eta_-]\, u + \{\eta_+(A_{-1}) - \eta_-(A_1) + A_1 A_{-1} - A_{-1} A_1\}\, u \\ &= \frac{i}{2}\{KD_V u + [\star F_\nabla, u] + \star(\nabla A + A \wedge A)u\} \end{aligned}$$

by 1 and Lemma 13.16. Then since

$$F_{\nabla^0} = F_\nabla + \nabla A + A \wedge A,$$

2 follows.

Finally to see 3, using the fact that $\mu_+^* = -\mu_-$ and $\mu_-^* = -\mu_+$, we have

$$\begin{aligned} \|\mu_+ u\|^2 &= \langle \mu_+ u, \mu_+ u\rangle_{L^2} \\ &= \left\langle \mu_+^* \mu_+ u, u\right\rangle_{L^2} \\ &= \langle -\mu_- \mu_+ u, u\rangle_{L^2} \\ &= \langle -\mu_+ \mu_- u, u\rangle_{L^2} + \langle [\mu_+, \mu_-]\, u, u\rangle_{L^2} \\ &= \langle \mu_-^* \mu_- u, u\rangle_{L^2} + \langle [\mu_+, \mu_-]\, u, u\rangle_{L^2} \\ &= \|\mu_- u\|^2 + \langle [\mu_+, \mu_-]\, u, u\rangle_{L^2}, \end{aligned}$$

and the result is then immediate from 2. □

13.5. A distance between transparent connections

In this section we introduce a metric $\mathbf{d}$ onto the space $\mathcal{T} = \mathcal{T}(E)$. Here we take (M, g) to be a closed Riemannian surface with negative curvature, and $p : E \to M$ a Hermitian complex vector bundle of rank n. To begin with, suppose ∇^1 and ∇^2 are two unitary connections on E, with $\nabla^2 = \nabla^1 + A$ for $A \in \Omega^1(M, \operatorname{ad} E)$. The induced connections $\pi^*\nabla^1$ and $\pi^*\nabla^2$ are related by

$$\pi^*\nabla^2 = \pi^*\nabla^1 + \pi^*A.$$

Thus the induced connections D^1 and D^2 on $\operatorname{End} \pi^*E$ are related by

$$D^2 = D^1 + \left[\pi^*A, \cdot\right].$$

The connections ∇^i induce π^*E-valued cocycles C_i over ϕ_t by

$$C_i(x, v, t) : (\pi^*E)_{(x,v)} \to (\pi^*E)_{\phi_t(x,v)}, \quad \xi \mapsto P^i_{\gamma_{(x,v)}}(t)(\xi),$$

where $P^i_{\gamma_{(x,v)}}$ is the parallel transport defined by ∇^i along the geodesic $\gamma_{(x,v)}$. If ∇^1 and ∇^2 both happen to be transparent, then the C_i satisfy the periodic orbit obstruction condition.

PROPOSITION 13.19. *Suppose both ∇^1 and ∇^2 are transparent. Then there exists a smooth function $u \in \Omega^0(SM, \operatorname{Aut} \pi^*E)$ such that*

$$C_2(x, v, t) = u(\phi_t(x, v)) \circ C_1(x, v, t) \circ u(x, v)^{-1}. \tag{13.5.1}$$

PROOF. Using Proposition 13.4 as before we deduce that π^*E is trivial. We thus obtain $U(n)$-valued cocycles $D_i : SM \times \mathbb{R} \to U(n)$ defined by

$$\Phi \circ C_i(x, v, t)(\Phi^{-1}(x, v, a)) = (\phi_t(x, v), D_i(x, v, t)a), \quad (x, v, a) \in SM \times \mathbb{C}^n.$$

where $\Phi : \pi^*E \to SM \times \mathbb{C}^n$ is a unitary trivialization. Then we can apply Theorem 6.12 to deduce the existence of smooth functions $u_i : SM \to U(n)$ such that

$$D_i(x, v, t) = u_i(\phi_t(x, v))u_i(x, v)^{-1} \quad \text{for all } (x, v, t) \in SM \times \mathbb{R}.$$

Now define a bundle automorphism u by

$$u(x, v) : (\pi^*E)_{(x,v)} \to (\pi^*E)_{(x,v)};$$
$$u(x, v)\xi := \Phi^{-1}\left((x, v), u_2(x, v)u_1(x, v)^{-1}(\mathrm{pr}_2 \circ \Phi(\xi))\right),$$

where $\mathrm{pr}_2 : SM \times \mathbb{C}^n \to \mathbb{C}^n$ is the second projection. Unraveling the definitions, we have

$$u(\phi_t(x, v)) \circ C_1((x, v), t) \circ u(x, v)^{-1}\xi = C_2((x, v), t)\xi,$$

which completes the proof. □

From this we can prove:

PROPOSITION 13.20. *It holds that*

$$D^1_X u + Au = 0.$$

PROOF. Fix $(x, v) \in SM$ and take $\xi \in (\pi^*E)_{(x,v)}$. Since the curve $t \mapsto C_i(x, v, t)\xi$ is $\pi^*\nabla^i$-parallel, if s is the section along $t \mapsto \phi_t(x, v)$ defined by

$$s(t) = C_2(x, v, t)\xi$$

then

$$\begin{aligned}
(\pi^*\nabla^1)C_2(x, v, t)\xi\big|_{t=0} &= (\pi^*\nabla^1)s(0) \\
&= (\pi^*\nabla^2 - \pi^*A)s(t)\big|_{t=0} \\
&= 0 - (\pi^*A)s(0) \\
&= -(\pi^*A)\xi.
\end{aligned}$$

Similarly

$$(\pi^*\nabla^1)u\left(\phi_t(x, v)C_1(x, v, t)(u(x, v)^{-1}\xi)\right)\big|_{t=0} = D^1u(u^{-1}\xi),$$

and hence we have shown that applying $\pi^*\nabla^1$ to both sides of (13.5.1), setting $t = 0$ and evaluating on X we obtain

$$-A\xi = D^1_X u(u^{-1}\xi).$$

Using the fact that (x, v) and ξ were arbitrary, the conclusion follows. □

Given $u \in \Omega^0(SM, \operatorname{End} \pi^* E)$, we can write

$$u = \sum_{m \in \mathbb{Z}} u_m,$$

where

$$u_m = \operatorname{proj}_{\Omega_m} \circ u \in \Omega_m.$$

Let us say that u has *degree* $N \in \mathbb{N}$ if $u_m = 0$ for all $|m| > N$, and N is minimal with this property. If no such N exists, we say u has degree ∞.

The crucial fact we wish to prove is that if u solves

$$D_X^1 u + Au = 0 \tag{13.5.2}$$

then u necessarily has *finite* degree. More precisely, we have the following result.

THEOREM 13.21. *Let $u \in \Omega^0(SM, \operatorname{End} \pi^* E)$ solve* (13.5.2). *Then u has finite degree.*

In fact, we have the following explicit upper bound on the degree: if $l \in \mathbb{N}$ is such that the Hermitian operators

$$\Sigma_{l,x}^+ : \operatorname{End} E_x \to \operatorname{End} E_x, \quad \alpha \mapsto -lK(x)\alpha + (i \star F_{\nabla^2}(x))\,\alpha - \alpha\,(i \star F_{\nabla^1}(x))$$

and

$$\Sigma_{l,x}^- : \operatorname{End} E_x \to \operatorname{End} E_x, \quad \alpha \mapsto -lK(x)\alpha - (i \star F_{\nabla^2}(x))\,\alpha + \alpha\,(i \star F_{\nabla^1}(x))$$

are positive definite for all $x \in M$ then the degree N of u satisfies $N < l$.

PROOF. Since $D_X = \eta_+ + \eta_-$, (13.5.2) may be rewritten as

$$\mu_+(u) + \mu_-(u) = 0.$$

Projecting onto Ω_m-components we obtain

$$\mu_+(u_{m-1}) + \mu_-(u_{m+1}) = 0 \tag{13.5.3}$$

for all $m \in \mathbb{Z}$. Since $K < 0$, there exists a positive integer l such that the Hermitian operators

$$u \mapsto -l\,Ku + (i \star F_{\nabla^2})u - u(i \star F_{\nabla^1}),$$
$$u \mapsto -l\,Ku - (i \star F_{\nabla^2})u + u(i \star F_{\nabla^1})$$

are positive definite for all $x \in M$. Using Lemma 13.18.3, we can find a constant $c > 0$ such that

$$\|\mu_+(u_m)\|^2 \geq \|\mu_-(u_m)\|^2 + c\,\|u_m\|^2 \tag{13.5.4}$$

for all $m \geq l$. There is also a constant $d > 0$ such that

$$\|\mu_-(u_m)\|^2 \geq \|\mu_+(u_m)\|^2 + d\,\|u_m\|^2 \tag{13.5.5}$$

for all $m \leq -l$. Combining (13.8.2) and (13.5.4) we obtain

$$\|\mu_+(u_{m+1})\| \geq \|\mu_+(u_{m-1})\| \tag{13.5.6}$$

for all $m \geq l - 1$. Similarly, it follows from (13.8.2) and (13.5.5) that

$$\|\mu_-(u_{m-1})\| \geq \|\mu_-(u_{m+1})\| \tag{13.5.7}$$

for all $m \geq -l + 1$. Since the function u is smooth, $\mu_+(u_m)$ must tend to zero in the L^2-topology as $m \to \infty$. It follow from (13.5.6) that $\mu_+(u_m) = 0$ for $m \geq l - 2$. However, (13.5.4) implies that μ_+ is injective for $m \geq l$ and thus $u_m = 0$ for $m \geq l$. Similarly, using (13.5.5) and (13.5.7) we deduce that $u_m = 0$ for $m \leq -l$. This shows that u has finite degree $N \leq l - 1$. □

With these auxiliary results now complete, we can get on to stating and proving the first main result of the section.

THEOREM 13.22. *Let (M, g) be a Riemannian surface of negative curvature and $p : E \to M$ a Hermitian vector bundle of rank n over M. Let $\mathcal{T}$ denote the space of transparent connections on E modulo gauge equivalence. Given $\alpha, \beta \in \mathcal{T}$, choose representatives $\nabla^1 \in \alpha$, $\nabla^2 \in \beta$ and define $\mathbf{d}(\alpha, \beta) = N$ where N is the smallest degree of an element $u \in \Omega^0(SM, \operatorname{Aut} \pi^* E)$ such that*

$$D_X^1 u + Au = 0,$$

where $\nabla^2 = \nabla^1 + A$, $A \in \Omega^1(M, \operatorname{ad} E)$ and D^1 is the connection on $\operatorname{End} \pi^* E$ induced by ∇^1.

Then $\mathbf{d}$ defines a metric on $\mathcal{T}$.

PROOF. First we check that $\mathbf{d}$ is well defined. Suppose $\omega, \nu \in \mathcal{G}_E$. Write $\nabla^1 = d + A_1$ and $\nabla^2 = d + A_2$, so $A = A_2 - A_1$.

Now we have

$$D^1 u = du + [\pi^* A_1, u],$$

and hence

$$\begin{aligned} D^1_X u &= du(X) + [A_1, u] \\ &= du(X) + A_1 u - u A_1. \end{aligned}$$

By definition,

$$\nu^* \nabla^1 = d + \nu^{-1} d\nu + \nu^{-1} A_1 \nu;$$
$$\omega^* \nabla^2 = d + \omega^{-1} d\omega + \omega^{-1} A_2 \omega.$$

Thus $\omega^* \nabla^2 - \nu^* \nabla^1 = \alpha$, where

$$\begin{aligned} \alpha &= \omega^{-1} d\omega + \omega^{-1} A_2 \omega - \nu^{-1} d\nu - \nu^{-1} A_1 \nu \\ &= \omega^{-1} A \omega + \omega^{-1} \nabla^1 \omega - \nu^{-1} \nabla^1 \nu. \end{aligned}$$

Write D^ν for the connection $(\pi^*(\nu^* \nabla^1))^{\mathrm{End}}$ on $\mathrm{End}\, \pi^* E$. Then

$$D^\nu = d + \left[\pi^*(\nu^{-1} d\nu + \nu^{-1} A_1 \nu), \cdot\right],$$

and hence

$$D^\nu_X u = du(X) + \nu^{-1} d\nu[X] u + \nu^{-1} A_1(X) \nu u - u \nu^{-1} d\nu(X) - u \nu^{-1} A_1(X) \nu.$$

Thus

$$\begin{aligned} D^\nu_X(\omega^{-1} u \nu) + \alpha(\omega^{-1} u \nu) &= d(\omega^{-1} u \nu)(X) + \nu^{-1} d\nu(X) \omega^{-1} u \nu \\ &\quad + \nu^{-1} A_1(X) \nu \omega^{-1} u \nu - \omega^{-1} u \nu \nu^{-1} d\nu(X) \\ &\quad - \omega^{-1} u \nu \nu^{-1} A_1(X) \nu + \omega^{-1} A_2(X) \omega \omega^{-1} u \nu \\ &\quad + \omega^{-1} d\omega(X) \omega^{-1} u \nu - \nu^{-1} d\nu(X) \omega^{-1} u \nu \\ &\quad - \nu^{-1} A_1(X) \nu \omega^{-1} u \nu. \end{aligned}$$

Expanding $d(\omega^{-1} u \nu)(X)$ we find that every term cancels and we are left with

$$D^\nu_X(\omega^{-1} u \nu) + \alpha(\omega^{-1} u \nu) = \omega^{-1}(D^1_X u + Au)\nu.$$

Thus we have

$$D^\nu_X(\omega^{-1} u \nu) + \alpha(\omega^{-1} u \nu) = 0 \quad \Leftrightarrow \quad D^1_X u + Au = 0.$$

The degree of u is equal to the degree of $\omega^{-1} u \nu$ since both ω and ν are elements of $\Omega^1(M, \mathrm{Aut}\, E)$, and hence this shows that $\mathbf{d}(\alpha, \beta)$ is independent of the choice of transparent connections ∇^1 and ∇^2 in the gauge equivalence class α and β respectively.

It thus remains to see that $\mathbf{d}$ is actually a metric. Suppose $\mathbf{d}(\alpha, \beta) = 0$. This means that there exists $u \in \Omega^0(SM, \mathrm{Aut}\, \pi^* E)$ solving

$$D^1_X u + Au = 0 \quad \text{and} \quad D^1_V u = 0.$$

But

$$D^1_V u = du(V) + [A_1(d\pi(V)), u] = V(u),$$

and $V(u) = 0$ implies that $u = \omega \circ \pi$ for some $\omega \in \Omega^0(M, \mathrm{Aut}\, \pi^* E)$. Let us write $\widehat{\nabla}^i$ for the connection $(\nabla^i)^{\mathrm{End}}$. We have

$$\begin{aligned} D^1_X u(x, v) &= (D^1_X \omega \circ \pi)(x, v) \\ &= \widehat{\nabla}^1_{d\pi(X)}(\omega \circ \pi)(x, v) \\ &= \widehat{\nabla}^1_v \omega, \end{aligned}$$

and so $D^1_X u + Au = 0$ implies $\widehat{\nabla}^1 \omega + A\omega = 0$. Since

$$\widehat{\nabla}^2 \omega = \widehat{\nabla}^1 \omega + [A, \omega],$$

we have

$$\widehat{\nabla}^2 \omega = -\omega A,$$

that is,

$$\begin{aligned}-\omega(\nabla^2 s)-\omega(\nabla^1 s) &= -\omega A(s)\\ &= \widehat{\nabla}^2\omega(s)\\ &= \nabla^2(\omega(s))-\omega(\nabla^2 s),\end{aligned}$$

that is,

$$\nabla^1(s)=\omega^{-1}(\nabla^2(\omega(s))),$$

or $\nabla^1=\omega^*\nabla^2$. Thus $\mathbf{d}(\alpha,\beta)=0$ implies $\alpha=\beta$.

To see $\mathbf{d}(\alpha,\beta)=\mathbf{d}(\beta,\alpha)$, we claim that if u solves $D^1_X u+Au=0$ then the adjoint u^* solves $D^2_X u^*-Au^*=0$. Since clearly u has the same degree as u^*, this would prove symmetricity. To see this note that $D^1_X u+Au=0$ implies

$$D^2_X u^*-Au^*=D^1_X u^*-u^*A=(D^1_X u+Au)^*.$$

Finally take a class $\gamma\in\mathcal{T}$ and $\nabla^3\in\gamma$. Write $\nabla^3=\nabla^1+B$ and suppose $D^1_X u+Au=0$ and $D^1_X w+Bw=0$ for $u,w\in\Omega^0(SM,\operatorname{End}\pi^*E)$. Using $\nabla^3=\nabla^2+(B-A)$ and $D^2=D^1+[\pi^*A,\cdot]$ and also that $D^1_X u^*=u^*A$ we compute that

$$\begin{aligned}D^2_X(wu^*) &= (D^2_X w)u^*+w(D^2_X u^*)\\ &= (D^1_X w+Aw-wA)u^*+w(D^1_X u^*+Au^*-u^*A)\\ &= (A-B)wu^*,\end{aligned}$$

and since $\deg(wu^*)\le\deg(w)+\deg(u)$ we see that $\mathbf{d}(\gamma,\beta)\le\mathbf{d}(\gamma,\alpha)+\mathbf{d}(\alpha,\beta)$. This completes the proof. □

The next corollary states that in the trivial bundle the trivial connection is locally unique among transparent connections.

COROLLARY 13.23. *Let (M,g) be a closed surface with curvature $K<0$. Let $E=M\times\mathbb{C}^n$ be the trivial bundle over M, and suppose ∇ is a transparent connection on E. Suppose the Hermitian matrix*

$$\pm i\star F_\nabla(x)-K(x)\mathrm{Id}$$

is positive definite for all $x\in M$. Then ∇ is gauge equivalent to the trivial connection d.

PROOF. By Theorem 13.21, if α denotes the gauge equivalence class of ∇ and $\alpha_0\in\mathcal{T}$ denotes the equivalence class of d then $\mathbf{d}(\alpha,\alpha_0)=0$. Thus ∇ is gauge equivalent to d, since $\mathbf{d}$ is a metric. □

13.6. Classifying transparent connections on trivial bundles

In this section we will explain and sketch the proof of [**Pat09a**, Theorem 3.1] (see also [**Pat09b**, Theorem B]), which gives an explicit description of all the transparent connections over the trivial vector bundle for (M,g) a closed negatively curved surface. We work with E_0 the trivial bundle $E_0=M\times\mathbb{C}^n$ only, which has the advantage of simplifying the notation somewhat, as well as the statements of the results. The main result in this section is Corollary 13.29, which is derived as an immediate consequence of the more general Theorem 13.28 below. It is still possible to prove a statement like Corollary 13.29 for an arbitrary Hermitian vector bundle E (modulo an assumption on $c_1(E)$, c.f. Proposition 13.7). For this we refer the reader to [**Pat09b**, Theorem B]. As before, we assume throughout that (M,g) is a closed Riemannian surface, and $\{X,H,V\}$ is the canonical framing of SM. Define

$$\mathcal{A}:=\left\{A:SM\to\mathfrak{u}(n)\ :\ V^2(A)=-A\right\}.$$

EXERCISE 13.24. Show that there is a bijective correspondence between $\mathcal{A}$ and the set of all unitary connections on E_0.

Recall that for each unitary connection ∇ on E_0 there is a natural cocycle C over ϕ_t. Let us now say a connection ∇ on E_0 is *cohomologically trivial* if the associated cocycle C is cohomologically trivial. Then a cohomologically trivial connection is always transparent, and by Theorem 13.9 if ϕ_t is Anosov (e.g. (M,g) has negative curvature) then ∇ is cohomologically trivial if and only if ∇ is transparent.

Let $\mathcal{A}_0\subseteq\mathcal{A}$ denote the set of functions $A:SM\to\mathfrak{u}(n)$ that correspond to cohomologically trivial connections.

EXERCISE 13.25. Show that $A \in \mathcal{A}_0$ if and only if there exists a smooth function $u : SM \to U(n)$ such that $X(u) + Au = 0$ and $V^2(A) = -A$.

We wish to study cohomologically trivial connections up to gauge equivalence. To do this, we first introduce a particular nonlinear PDE for functions $f : SM \to \mathfrak{u}(n)$:

$$H(f) + VX(f) = [X(f), f]. \tag{13.6.1}$$

Let $\mathcal{H} \subseteq C^\infty(SM, \mathfrak{u}(n))$ denote the set of solutions to (13.6.1). Note that if $u \in U(n)$ and $f \in \mathcal{H}$ then $u^{-1} f u \in \mathcal{H}$. Given two solutions f, h of (13.6.1), we say f and h are *V-cohomologous* if there exists $u : SM \to U(n)$ such that

$$f = u^{-1} V(u) + u^{-1} h u.$$

Let $\mathcal{H}_0 \subseteq \mathcal{H}$ denote the set of solutions that are V-cohomologous to zero, that is

$$\mathcal{H}_0 := \left\{ f \in \mathcal{H} \,:\, \text{there exists } u : SM \to U(n) \text{ such that } f = u^{-1} V(u) \right\}.$$

Finally given any vector field Y on SM, let $\mathcal{G}_Y \subseteq C^\infty(SM, U(n))$ denote the set of all functions $u : SM \to U(n)$ such that $Y(u) = 0$. Note that

$$\mathcal{G}_V = \mathcal{G}_{E_0}$$

is the gauge group of the trivial bundle $M \times \mathbb{C}^n$.

LEMMA 13.26. *The set $\mathcal{G}_X$ acts on $\mathcal{H}_0$ via*

$$(a, f) \in \mathcal{G}_X \times \mathcal{H}_0 \mapsto a \cdot f := a^{-1} f a + a^{-1} V(a) \in \mathcal{H}_0.$$

EXERCISE 13.27. Prove the lemma.

Here is the main theorem of this section.

THEOREM 13.28. *There is a bijective correspondence between $\mathcal{A}_0/\mathcal{G}_V$ and $\mathcal{H}_0/\mathcal{G}_X$.*

COROLLARY 13.29. *Suppose (M, g) has negative curvature. Then there is a bijective correspondence between the set $\mathcal{T}$ of transparent connections on E_0 and $\mathcal{H}_0/U(n)$.*

EXERCISE 13.30. Assuming Theorem 13.28, prove the Corollary 13.29 *(hint: what is $\mathcal{G}_X$ when ϕ_t is transitive?).*

PROOF. *(of Theorem 13.28)*

Let us start with a cohomologically trivial connection ∇ on E_0. This is the same as an element $A \in \mathcal{A}_0$, and the fact that A lies in $\mathcal{A}_0$ tells us that there exists $u \in C^\infty(SM, U(n))$ with $X(u) + Au = 0$. Set

$$f := u^{-1} V(u).$$

Then we claim that f satisfies the PDE (13.6.1). We argue as follows. Using u we may define a connection $\bar{\nabla}$ on SM gauge equivalent to $\pi^*\nabla$ by setting

$$B := u^{-1} du + u^{-1} \pi^* A u$$

and $\bar{\nabla} = d + B$. Note that $f = B(V)$. Since $F_{\pi^*\nabla}(\cdot, V) = 0$, we must also have $F_{\bar{\nabla}}(\cdot, V) = 0$. Using the fact that $F_{\bar{\nabla}} = dB + B \wedge B$ and the fact that $B(X) = 0$ we compute

$$F_B(X, V) = dB(X, V) + [B(X), B(V)] = dB(X, V).$$

But

$$dB(X, V) = XB(V) - VB(X) - B([X, V]) = XB(V) + B(H),$$

and hence

$$B(H) = -XB(V). \tag{13.6.2}$$

We also compute

$$F_B(H, V) = dB(H, V) + [B(H), B(V)],$$

and combine this with

$$dB(H, V) = HB(V) - VB(H) - B([H, V]) = HB(V) - VB(H)$$

to obtain

$$HB(V) - VB(H) + [B(H), B(V)] = 0. \tag{13.6.3}$$

Combining (13.6.2) and (13.6.3) we see that f satisfies (13.6.1), as claimed. To see that f is unique up to an element of $\mathcal{G}_X$, note that if $f' = v^{-1}V(v)$ is another solution then $X(u^{-1}v) = 0$, i.e. $u^{-1}v \in \mathcal{G}_X$.

This proves one direction. Now suppose we are given a solution f of (13.6.1) such that there is $u : SM \to U(n)$ with with $f = u^{-1}V(u)$. Set

$$A := -X(u)u^{-1}.$$

We claim that $A \in \mathcal{A}_0$, that is, $V^2(A) = -A$. Indeed, first compute that

$$\begin{aligned} V(A) &= -V(X(u)u^{-1}) \\ &= -VX(u)u^{-1} - X(u)V(u^{-1}) \\ &= -XV(u)u^{-1} - H(u)u^{-1} - X(u)V(u^{-1}) \\ &= -X(uf)u^{-1} - H(u)u^{-1} - X(u)fu^{-1} \\ &= -uX(f)u^{-1} - H(u)u^{-1}. \end{aligned}$$

Now apply V again and use the fact that f satisfies (13.6.1) to verify $V^2(A) = -A$.

EXERCISE 13.31. Complete the details here.

To see that A is unique up to an element of $\mathcal{G}_V = \mathcal{G}_E$, observe that if we have another element v such that $f = v^{-1}V(v)$, this gives rise to a different connection $\nabla' = d + A'$, where $A' = -X(v)v^{-1}$, and that ∇ is gauge equivalent to ∇'. Indeed, if $w := uv^{-1}$ then

$$A' = w^{-1}dw + w^{-1}Aw.$$

Finally, it is clear that our two correspondences are mutually inverse; this completes the proof. □

13.7. The Bäcklund transformation

For the remainder of this chapter we restrict to the case in which the structure group is $SU(2)$. Suppose there is a smooth map $b : SM \to SU(2)$ such that $f := b^{-1}V(b)$ solves the PDE (13.6.1). Then, by Theorem 13.28, $A := -X(b)b^{-1}$ defines a cohomologically trivial connection on M and $- \star A = V(A) = -bX(f)b^{-1} - H(b)b^{-1}$.

LEMMA 13.32. *Let $g : M \to \mathfrak{su}(2)$ be a smooth map with $\det g = 1$ (i.e. $g^2 = -\mathrm{Id}$). Then there exists $a : SM \to SU(2)$ such that $g = a^{-1}V(a)$.*

PROOF. Let $L(x)$ (resp. $U(x)$) be the eigenspace corresponding to the eigenvalue i (resp. $-i$) of $g(x)$. We have an orthogonal decomposition $\mathbb{C}^2 = L(x) \oplus U(x)$ for every $x \in M$. Consider sections $\alpha \in \Omega^{1,0}(M, \mathbb{C})$ and $\beta \in \Omega^{1,0}(M, \mathrm{Hom}(L, U)) = \Omega^{1,0}(M, L^*U)$ such that $|\alpha|^2 + |\beta|^2 = 1$. Such a pair of sections always exists; for example, we can choose a section $\tilde{\beta}$ with a finite number of isolated zeros and then choose $\tilde{\alpha}$ such that it does not vanish on the zeros of $\tilde{\beta}$. Then we set $\alpha := \tilde{\alpha}/(|\tilde{\alpha}|^2 + |\tilde{\beta}|^2)^{1/2}$ and $\beta := \tilde{\beta}/(|\tilde{\alpha}|^2 + |\tilde{\beta}|^2)^{1/2}$. Note that $\bar{\alpha} \in \Omega^{0,1}(M, \mathbb{C})$ and $\beta^* \in \Omega^{0,1}(M, \mathrm{Hom}(U, L)) = \Omega^{0,1}(M, U^*L)$. Using the orthogonal decomposition we define $a : SM \to SU(2)$ by

$$a(x, v) = \begin{pmatrix} \alpha(x, v) & \beta^*(x, v) \\ -\beta(x, v) & \bar{\alpha}(x, v) \end{pmatrix}.$$

Clearly $a = a_{-1} + a_1$, where

$$a_1 = \begin{pmatrix} \alpha & 0 \\ -\beta & 0 \end{pmatrix}$$

and

$$a_{-1} = \begin{pmatrix} 0 & \beta^* \\ 0 & \bar{\alpha} \end{pmatrix}.$$

It is straightforward to check that $ag = V(a)$.

□

Now let $u := ab : SM \to SU(2)$ and let $F := (ab)^{-1}V(ab) = b^{-1}g\,b + f$.

Question. When does F satisfy (13.6.1)?

If it does, then it defines (via Theorem 13.28) a new cohomologically trivial connection $\nabla_F = d + A_F$ where A_F is given by

$$A_F = -X(ab)(ab)^{-1} = -X(a)a^{-1} + aAa^{-1},$$

and where A is the cohomologically trivial connection associated to f.

Write $\nabla = d + A$.

LEMMA 13.33. *F satisfies* (13.6.1) *if and only if*

$$-\star\nabla g = (\nabla g)g. \tag{13.7.1}$$

PROOF. Starting with $F = b^{-1}gb + f$ and using that $A = -X(b)b^{-1} = bX(b^{-1})$ we compute

$$X(F) = b^{-1}([A, g] + X(g))b + X(f).$$

Similarly, using $H(b) = (\star A)b - bX(f)$ we find

$$H(F) = b^{-1}([-\star A, g] + H(g))b + [X(f), b^{-1}g\,b] + H(f).$$

Now we compute $VX(F)$; here we use that $V(g) = 0$. We obtain

$$VX(F) = [b^{-1}([A, g] + X(g))b, f] + b^{-1}([-\star A, g] + VX(g))b + VX(f).$$

The last term we need for (13.6.1) is:

$$[X(F), F] = b^{-1}[[A, g] + X(g), g]b + [b^{-1}([A, g] + X(g))b, f] + [X(f), b^{-1}g\,b] + [X(f), f].$$

Since f satisfies (13.6.1) we see that F satisfies (13.6.1) if and only if

$$H(g) + VX(g) - 2[\star A, g] = [[A, g] + X(g), g].$$

Since g depends only on the base point and $g^2 = -\mathrm{Id}$ we can rewrite this as

$$-2\star(dg + [A, g]) = [dg + [A, g], g] = 2(dg + [A, g])\,g.$$

Thus F satisfies (13.6.1) if and only if

$$-\star\nabla g = (\nabla g)g$$

as claimed.

□

We will now rephrase equation (13.7.1) in terms of holomorphic line bundles. Let us write $\mathbb{M}_2(\mathbb{C})$ for the set of all 2×2 complex matrices.Recall that the connection A induces a holomorphic structure on the trivial bundle $M \times \mathbb{C}^2$ and on the endomorphism bundle $M \times \mathbb{M}_2(\mathbb{C})$. We have an operator

$$\bar\partial_A := (\nabla - i\star\nabla)/2 = \bar\partial + [A_{-1}, \cdot]$$

acting on sections $f : M \to \mathbb{M}_2(\mathbb{C})$. Set $\pi := (\mathrm{Id} - ig)/2$ and $\pi^\perp = (\mathrm{Id} + ig)/2$ so that $\pi + \pi^\perp = \mathrm{Id}$. Let $L(x)$ be as above the eigenspace corresponding to the eigenvalue i of $g(x)$. Note that π is the Hermitian orthogonal projection over $L(x) = \mathrm{Im}\,(\pi(x))$.

LEMMA 13.34. *Let $g : M \to \mathfrak{su}(2)$ be a smooth map with* $\det\, g = 1$. *The following are equivalent:*

(1) $-\star\nabla g = (\nabla g)g$;
(2) *L is a $\bar\partial_A$-holomorphic line bundle;*
(3) $\pi^\perp\bar\partial_A\pi = 0$.

PROOF. Suppose that (1) holds. Apply $\star$ to obtain: $\nabla g = (\star\nabla g)g$. Thus $\nabla g - i\star\nabla g = i(\nabla g - i\star\nabla g)g$. In other words $\bar\partial_A g = i(\bar\partial_A g)g = -ig(\bar\partial_A g)$ (recall that $g^2 = -\mathrm{Id}$). Since $\pi = (\mathrm{Id} - ig)/2$, then $\bar\partial_A g = -ig(\bar\partial_A g)$ is equivalent to $\pi^\perp\bar\partial_A\pi = 0$ which is (3).

Using the condition $\pi^2 = \pi$, we see that $\pi^\perp\bar\partial_A\pi = 0$ is equivalent to $(\bar\partial_A\pi)\pi = 0$. The line bundle L is holomorphic if and only if given a local section ξ of L, then $\bar\partial_A\xi \in L$. Using that $\pi\xi = \xi$ we see that $\bar\partial_A\xi \in L$ if and only if $(\bar\partial_A\pi)\xi = 0$. Clearly, this happens if and only if $(\bar\partial_A\pi)\pi = 0$ and thus (2) holds if and only if (3) holds.

□

The next theorem follows directly from Lemmas 13.33 and 13.34 and Theorem 13.28.

THEOREM 13.35. *Let A be a cohomologically trivial connection and let L be a holomorphic line subbundle of the trivial bundle $M \times \mathbb{C}^2$ with respect to the complex structure induced by A. Define a map $g : M \to \mathfrak{su}(2)$ with* $\det g = 1$ *by declaring L to be its eigenspace with eigenvalue i. Consider $a : SM \to SU(2)$ with $g = a^{-1}V(a)$ as given by Lemma 13.32. Then*

$$A_F := -X(a)a^{-1} + aAa^{-1}$$

defines a cohomologically trivial connection.

REMARK 13.36. If the geodesic flow is transitive, two solutions u, w of $X(u) + Au = 0$ are related by $u = wg$ where g is a constant unitary matrix, because $X(w^{-1}u) = 0$. Thus the degrees of u and w are the same. We can then talk about the "degree" of a cohomologically trivial connection as the degree of any solution of $X(u) + Au = 0$.

If we start, for example, with the trivial connection $A = 0$ (which is obviously transparent), then a map $g : M \to \mathfrak{su}(2)$ with $\det g = 1$ and $-\star dg = (dg)g$ can be identified with a meromorphic function. The point of Theorem 13.35 is to show how to increase the degree of a cohomologically trivial connection by one via $A \mapsto A_F$. We shall call this transformation a *Bäcklund transformation*. In the next section we will show how we can also run this procedure "backwards" to decrease the degree of a transparent connection. We will thus show that any cohomologically trivial connection such that the associated u has a finite Fourier series can be built up by successive applications of the transformation described in Theorem 13.35, provided that the geodesic flow is transitive.

13.8. Lowering degree using Bäcklund transformations

Let A be a transparent connection with $A = -X(b)b^{-1}$ and $f = b^{-1}V(b)$, where $b : SM \to SU(2)$, as in the previous section.

We first make some remarks concerning the $SU(2)$-structure. Let $\mathtt{j} : \mathbb{C}^2 \to \mathbb{C}^2$ be the antilinear map given by

$$\mathtt{j}(z_1, z_2) = (-\bar{z}_2, \bar{z}_1).$$

If we think of a matrix $a \in SU(2)$ as a linear map $a : \mathbb{C}^2 \to \mathbb{C}^2$, then $\mathtt{j}a = a\mathtt{j}$. This implies that given $b : SM \to SU(2)$ with $b = \sum_{k\in\mathbb{Z}} b_k$, then $\mathtt{j}b_k = b_{-k}\mathtt{j}$ for all $k \in \mathbb{Z}$.

Assumption. Suppose b has a finite Fourier expansion, i.e., $b = \sum_{k=-N}^{k=N} b_k$, where $N \geq 1$. By Theorem 13.21 we know that this holds if M has negative curvature.

Let us assume also that N is the degree of b and thus both b_N and $b_{-N} = -\mathtt{j}b_N\mathtt{j}$ are non-zero.

The unitary condition $bb^* = b^*b = \mathrm{Id}$ implies that $b_N b_{-N}^* = b_{-N}^* b_N = 0$. These relations imply that the rank of b_{-N} and b_N is at most one and equals one on an open set, which, as we will see shortly, must be all of M except for perhaps a finite number of points.

We will now use the operators $\eta_\pm$ and $\mu_\pm$ from Section 13.4 again. Consider isothermal coordinates (x^1, x^2) on M such that the metric can be written as $ds^2 = e^{2\lambda}(d(x^1)^2 + d(x^2)^2)$ where λ is a smooth real-valued function of (x^1, x^2) (see Chapter 4). This gives coordinates (x^1, x^2, θ) on SM where θ is the angle between a unit vector v and $\partial/\partial x$. In these coordinates, $V = \partial/\partial\theta$ and the vector fields X and H are given by (4.1.4) and (4.1.5). Consider $u \in \Omega_n$ and write it locally as $u(x^1,{}^2, \theta) = h(x^1, x^2)e^{in\theta}$. Using these formulas a simple, but tedious calculation shows that

$$\eta_-(u) = e^{-(1+n)\lambda}\bar{\partial}(he^{n\lambda})e^{i(n-1)\theta}, \tag{13.8.1}$$

where $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)$.

EXERCISE 13.37. Verify (13.8.1).

In order to write μ_- suppose that $A(x^1, x^2, \theta) = a(x^1, x^2)\cos\theta + b(x^1, x^2)\sin\theta$. If we also write $A = A_{x^1}dx^1 + A_{x^2}dx^2$, then $A_{x^1} = ae^{\lambda}$ and $A_{x^2} = be^{\lambda}$. Let $A_{\bar{z}} := \frac{1}{2}(A_{x^1} + iA_{x^2})$. Using the definition of A_{-1} we derive

$$A_{-1} = \frac{1}{2}(a + ib)e^{-i\theta} = A_{\bar{z}}d\bar{z}.$$

Putting this together with (13.8.1) we obtain

$$\mu_-(u) = e^{-(1+n)\lambda}\left(\bar{\partial}(he^{n\lambda}) + A_{\bar{z}}he^{n\lambda}\right)e^{i(n-1)\theta}. \tag{13.8.2}$$

Note that Ω_n can be identified with the set of smooth sections of the bundle $(M \times \mathbb{M}(\mathbb{C})) \otimes K^{\otimes n}$ where K is the canonical line bundle. The identification takes $u = he^{in\theta}$ into $he^{n\lambda}(dz)^n$ $(n \geq 0)$ and $u = he^{-in\theta} \in \Omega_{-n}$ into $he^{n\lambda}(d\bar{z})^n$.

Consider now a fixed vector $\xi \in \mathbb{C}^2$ such that $s(x,v) := b_{-N}(x,v)\xi \in \mathbb{C}^2$ is not identically zero. Clearly s can be seen as a section of $(M\times\mathbb{C}^2)\otimes K^{\otimes -N}$. We may write b_{-N} in local isothermal coordinates as $b_{-N} = he^{-iN\theta}$. We can thus write s locally as $s = e^{N\lambda}h\xi(d\bar{z})^N$.

LEMMA 13.38. *The local section $e^{-2N\lambda}s$ is $\bar{\partial}_A$-holomorphic.*

PROOF. Using the operators $\mu_\pm$ we can write $X(b) + Ab = 0$ as

$$\mu_+(b_{k-1}) + \mu_-(b_{k+1}) = 0$$

for all k. This gives $\mu_+(b_N) = \mu_-(b_{-N}) = 0$. But $\mu_-(b_{-N}) = 0$ is saying that $e^{-2N\lambda}s$ is $\bar{\partial}_A$-holomorphic. Indeed, using (13.8.2), we see that $\mu_-(b_{-N}) = 0$ implies

$$\bar{\partial}(he^{-N\lambda}) + A_{\bar{z}}he^{-N\lambda} = 0$$

which in turn implies

$$\bar{\partial}(e^{-N\lambda}h\xi) + A_{\bar{z}}e^{-N\lambda}h\xi = 0.$$

This equation says that $e^{-2N\lambda}s = e^{-N\lambda}h\xi(d\bar{z})^N$ is $\bar{\partial}_A$-holomorphic.

□

The section s spans a line bundle L over M which by the previous lemma is $\bar{\partial}_A$-holomorphic. The section s may have zeros, but at a zero z_0, the line bundle extends holomorphically. Indeed, in a neighborhood of z_0 we may write $e^{-2N\lambda(z)}s(z) = (z-z_0)^k w(z)$, where w is a local holomorphic section with $w(z_0) \neq 0$. The section w spans a holomorphic line subbundle which coincides with the one spanned by s off z_0. Therefore L is a $\bar{\partial}_A$-holomorphic line bundle that contains the image of b_{-N} (and $U = \mathtt{j}L$ is an anti-holomorphic line bundle that contains the image of b_N). We summarize this in a lemma:

LEMMA 13.39. *The line bundle L determined by the image of b_{-N} is $\bar{\partial}_A$-holomorphic.*

We now wish to use the line bundle L to construct an appropriate $g : M \to \mathfrak{su}(2)$ such that when we run the Bäcklund transformation from the previous section we obtain a cohomologically trivial connection of degree $\leq N-1$. But first we need the following lemma. Recall that a matrix-valued function $f : SM \to \mathbb{M}_n(\mathbb{C})$ is said to be *odd* if $f(x,v) = -f(x,-v)$ and *even* if $f(x,v) = f(x,-v)$.

LEMMA 13.40. *Assume that the geodesic flow is transitive and let $b : SM \to SU(2)$ solve $X(b) + Ab = 0$. Then b is either even or odd.*

PROOF. Write $b = b_o + b_e$ where b_o is odd and b_e is even. Since the operator $(X + A)$ maps even to odd and odd to even, the equation $X(b) + Ab = 0$ decouples as

$$X(b_o) + Ab_o = 0;$$

$$X(b_e) + Ab_e = 0.$$

A calculation using these equations shows that $X(b_o^*b_o) = X(b_e^*b_e) = X(b_o^*b_e) = 0$. Since the geodesic flow is transitive, these matrices are all constant. Moreover, since $b_o^*b_e$ is odd it must be zero. On the other hand $\mathtt{j}b = b\mathtt{j}$ implies that $\mathtt{j}b_o = b_o\mathtt{j}$ and $\mathtt{j}b_e = b_e\mathtt{j}$, which in turn implies that both b_o and b_e cannot have rank 1. Putting all this together, we see that either b_o or b_e must vanish identically.

□

Suppose that the geodesic flow is transitive. By Lemma 13.40, $b = b_{-N} + d + b_N$, where d has degree $\leq N-2$. We now seek $a : SM \to SU(2)$ of degree one such that $u := ab$ has degree $\leq N-1$. For this we need $a_1 b_N = a_{-1}b_{-N} = 0$. We take a map $g : M \to \mathfrak{su}(2)$ with $\det g = 1$ such that its i eigenspace is L and its $-i$ eigenspace is U. By Lemmas 13.34 and 13.39, $-\star\nabla g = (\nabla g)g$ where $\nabla = d + A$. The construction of a with $ag = V(a)$ from Lemma 13.32 is precisely such that the kernel of a_{-1} is L and the kernel of a_1 is U, so the needed relations to lower the degree hold.

Finally by Theorem 13.35, u gives rise to a cohomologically trivial connection of the form $-X(u)u^{-1}$. Combining this with Proposition 13.21 we have arrived at the final result of these notes, which is from [**Pat09a**, Theorem 5.4].

THEOREM 13.41. *Let M be a closed orientable surface of negative curvature. Then any transparent $SU(2)$-connection can be obtained by successive applications of Bäcklund transformations as described in Theorem 13.35.*

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