1. Prove the Mittag – Leffler theorem: For any sequence of points (z_n) in $\mathbb{C}$ that converge to ∞ and any polynomials (p_n) (with zero constant terms), there is a meromorphic function $f : \mathbb{C} \to \mathbb{P}$ with poles only at the points z_n and the principal part of f at each z_n being $p_n((z-z_n)^{-1})$.
[*Write* $f(z) = \sum p_n(z) - q_n(z)$ *and choose the polynomials* q_n *so that the series converges locally uniformly.*]
2. Show that the product
$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$
converges and satisfies
$$g'(z) = g(z) \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$
Deduce that $g(z+1) = -zg(z)e^{\gamma}$ for some constant γ and prove that
$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log N.$$
(This is Euler's constant.)
3. A function $f : \mathbb{C} \to \mathbb{C}$ is *periodic with period* p if $f(z+p) = f(z)$ for every $z \in \mathbb{C}$. Show that the set of periods of a holomorphic function f is either a lattice in $\mathbb{C}$ or else all of $\mathbb{C}$.
4. Show that every holomorphic function $f : \mathbb{C} \to \mathbb{C}$ which is periodic with a period $p \neq 0$ has a Fourier expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n \exp(2\pi i n z/p)$ convergent everywhere.
5. Show that for any subset E of $\mathbb{C} \setminus \{0\}$ which has no accumulation points except possibly 0 or ∞ there is a holomorphic function on $\mathbb{C} \setminus \{0\}$ with zeros precisely at the points of E.
6. Show that $z \mapsto \omega(z - z_0)/(1 - \overline{z_0}z)$ is in $\mathrm{Aut}\,\mathbb{D}$ for ω with $|\omega| = 1$ and $z_o \in \mathbb{D}$. Conversely every map in $\mathrm{Aut}\,\mathbb{D}$ is of this form.
7. Find $\mathrm{Aut}\,\mathbf{H}^+$ for the upper half plane $\mathbf{H}^+ = \{z \in \mathbb{C} : \Im z > 0\}$.
8. Consider $\mathbb{D}$ as the subset
$$\{[z_1 : z_2] : |z_1|^2 - |z_2|^2 < 0\} \quad \text{of} \quad \mathbb{P}(\mathbb{C}^2).$$
Show that an invertible linear map $T : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ with determinant 1 induces a conformal map $\mathcal{T} : \mathbb{D} \to \mathbb{D}$ if, and only if,
   (a) T preserves the indefinite form
$$\beta : \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \mapsto \overline{z_1}w_1 - \overline{z_2}w_2.$$
   (That is $\beta(T\mathbf{z}, T\mathbf{w}) = \beta(\mathbf{z}, \mathbf{w})$.)
and
   (b) $\mathcal{T}(0) \in \mathbb{D}$. (That is $|b| < |d|$.)
9. Let $T \in \mathrm{Aut}\,\mathbb{D}$. Show that
   (a) if T is elliptic, it has exactly one fixed point in $\mathbb{D}$.
   (b) if T is hyperbolic, it has two fixed points both on $\partial\mathbb{D}$.
   (c) if T is parabolic, it has one fixed point on $\partial\mathbb{D}$.
Find the conjugacy classes in $\mathrm{Aut}\,\mathbb{D}$.
10. Prove directly that a loxodromic Möbius transformation cannot map any disc in $\mathbb{P}$ onto itself.
11. Show that the hyperbolic metric on $\mathbb{D}$ is complete.

12. (This requires the definition of a Riemannian metric.) Show that the only Riemannian metrics on $\mathbb{D}$ which are invariant under $\operatorname{Aut}\mathbb{D}$ are the multiples of the hyperbolic metric. Show that there are no Riemannian metrics on $\mathbb{P}$ or $\mathbb{C}$ which are invariant under $\operatorname{Aut}\mathbb{P}$ or $\operatorname{Aut}\mathbb{C}$.
13. Let z_1, z_2, w_1 and w_2 be four points in $\mathbb{D}$. Show that there is a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $f(z_1) = w_1$ and $f(z_2) = w_2$ if, and only if, $\rho(w_1, w_2) \leqslant \rho(z_1, z_2)$.
14. Let C be the unique circle through the two points $z_0, z_1 \in \mathbb{D}$ which is orthogonal to $\partial\mathbb{D}$. Then C meets $\partial\mathbb{D}$ at the points w_0, w_1 with w_0, z_0, z_1, w_1 in that order on C. Express the cross ratios of w_0, z_0, z_1, w_1 and of $J(z_0), z_0, z_1, J(z_1)$ in terms of $\rho(z_0, z_1)$.
[*Here* J *is inversion in the unit circle:* $J : z \mapsto 1/\overline{z}$.]
15. Prove that
$$\left|\frac{z_0 - z_1}{1 - \overline{z_0} z_1}\right| = \tanh \tfrac{1}{2}\rho(z_0, z_1).$$
Does the left side of this equation define a metric on $\mathbb{D}$? Find similar formulae for $\sinh \rho(z_0, z_1)$ and $\cosh \rho(z_0, z_1)$.
16. A Blaschke product on a finite set of points in $\mathbb{D}$ is called a *finite Blaschke product*. (This includes the constant maps $z \mapsto \omega$ for $|\omega| = 1$.) Prove that a continuous function $f : \overline{\mathbb{D}} \to \mathbb{C}$ is a finite Blashke product if, and only if, it is holomorphic on $\mathbb{D}$ and maps $\partial\mathbb{D}$ into itself.

    What are the continuous maps $f : \mathbb{D} \to \mathbb{P}$ which are meromorphic on $\mathbb{D}$ and map $\partial\mathbb{D}$ into itself?
17. Let B be the Blaschke product for a sequence (z_n) in $\mathbb{D}$ which satisfies $\sum 1 - |z_n| < \infty$. Show that the Blaschke product converges not only on $\mathbb{D}$ but also on $\{z \in \mathbb{P} : |z| > 1\}$ giving a meromorphic function with poles at the points $(J(z_n))$. Prove that $JB(z) = BJ(z)$ for $z \in \mathbb{D}$.

    If $z \in \partial\mathbb{D}$ is not the limit point of a sequence (z_n) then prove that the Blaschke product converges at z, is holomorphic on a neighbourhood, and satisfies $|B(z)| = 1$.
18. Let G be the group generated by a single Möbius transformation $T \in \text{Möb}(\mathbb{D})$. When is G a discrete group? In each case where it is a discrete group, describe a Dirichlet domain for G and the quotient $\mathbb{D}/G$.
19. Let G be a discrete subgroup of $\text{Möb}(\mathbb{D})$ and $z_o \in \mathbb{D}$ with trivial stablizer. Suppose that $w \in \mathbb{D}$ has a non-trivial stabilizer S of order N. Show that N points of the orbit Gw lie on the boundary of the Dirichlet domain $D(z_o)$. How many of the Dirichlet domains $D(T(z_o))$ for $T \in G$ meet at w?
20. Let D be a proper subdomain of the complex plane and (z_n) a sequence of points in D. The sequence (z_n) *converges to* ∂D if, for every compact subset K of D, only a finite number of the points z_n lie in K.

    Show that the zeros of a non-constant holomorphic function $f : D \to \mathbb{C}$ are either finite in number or else form a sequence that tends to ∂D.

    For $z_n \in D$, show that there is a point $w_n \in \mathbb{C} \setminus D$ with
$$|z_n - w_n| = \inf\{|z_n - w| : w \in \mathbb{C} \setminus D\}\ .$$
    Then
$$\frac{z - z_n}{z - w_n} = 1 - \left(\frac{z_n - w_n}{z - w_n}\right)$$
    and the power series for the principal branch
$$\log\left(\frac{z - z_n}{z - w_n}\right)$$
    converges uniformly on $\{z \in \mathbb{C} : |z - w_n| \geqslant 2|z_n - w_n|\}$. Deduce that a suitable product of terms
$$\left(\frac{z - z_n}{z - w_n}\right) \exp\left(\sum_{k=1}^{K_n} \frac{1}{k}\left(\frac{z_n - w_n}{z - w_n}\right)^k\right)$$
    converges locally uniformly on D to give a holomorphic function $f : D \to \mathbb{C}$ with zeros precisely at the points (z_n).